Kudla–Rapoport conjecture for Krämer models

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Theorem (Jacobi, 1820s)

Define representation number $r(n) := \#\{(x, y) \in \mathbb{Z}^2 : n = x^2 + y^2\}$. Then

$$r(n) = 4 \left(\sum_{\substack{d|n \\ d \equiv 1 \bmod 4}} 1 - \sum_{\substack{d|n \\ d \equiv 3 \bmod 4}} 1 \right)$$

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Example (Reprove Fermat)

- If $p \equiv 1 \pmod 4$, then $r(p) = 4(2 - 0) = 8$.
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Exercise: What is $r(n^2)$ for $n = 3 \cdot 13 \cdot 2023$?

A proof by modular forms

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- Jacobi's theta series for the quadratic form $x^2 + y^2$,

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- Summary of the proof: relate theta series and Eisenstein series.
- A simplest instance of the more general Siegel–Weil formula.

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- Weil (1960s): Weil representations ω for dual pairs of classical groups (G, H).
- F/F_0: quadratic extension of number fields. $\mathbb{A} = \mathbb{A}_{F_0}$: the ring of adeles of F_0.
- V: hermitian space over F of dimension m. $(G, H) = (\mathrm{U}(n, n), \mathrm{U}(V))$.

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$$\theta(g, h, \varphi) =: \sum_{x \in V^n} \omega(g, h)\varphi(x), \quad g \in G(\mathbb{A}), h \in H(\mathbb{A}).$$

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- Define Siegel Eisenstein series

$$E(g, s, \varphi) := \sum_{\gamma \in P \backslash G} \Phi_\varphi(\gamma g, s), \quad g \in G(\mathbb{A}), s \in \mathbb{C},$$

where $\mathscr{S}(V(\mathbb{A})^n) \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi_V, s), \quad \varphi \mapsto \Phi_\varphi(g, s) := \omega(g)\varphi(0) \cdot |a(g)|^s.$

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Theorem (Siegel–Weil formula (V anisotropic or $n \ll m$), 1960s)

$$\int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \theta(g, h, \varphi)\, \mathrm{d}h = E(g, s_0, \varphi), \quad s_0 = \frac{m - n}{2}.$$

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Remark

- Recover Jacobi's formula for $(G, H) = (\mathrm{Sp}(2), \mathrm{O}(2))$
- Extended to complete generality: [Kudla–Rallis, Ikeda, Ichino, Yamana, Gan–Qiu–Takeda, ...]

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- Assume F/F_0 is a CM extension of a totally real field. Fix $\sigma : F \hookrightarrow \mathbb{C}$.
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- $K \subseteq H(\mathbb{A}_f)$: open compact subgroup.
- X: unitary Shimura variety of dimension $m-1$ over $F \subseteq \mathbb{C}$ with complex uniformization

$$X(\mathbb{C}) = H(F_0)\backslash[\mathbb{D} \times H(\mathbb{A}_f)/K],$$

$$\mathbb{D} = \{\text{negative } \mathbb{C}\text{-lines in } V \otimes_F \mathbb{C}\} \simeq \{z \in \mathbb{C}^{m-1} : |z| < 1\} \simeq \frac{\mathrm{U}(m-1,1)}{\mathrm{U}(m-1)\times \mathrm{U}(1)}.$$

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- Motivic L-function of X should be factorized into a product of automorphic L-functions for $H(\mathbb{A})$ [Langlands, Kottwitz]. When V is standard indefinite, the L-function appearing should be the standard L-functions.

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- X is a Shimura variety of abelian type. Its étale cohomology and L-function are computed in forthcoming [Kisin–Shin–Zhu], under the help of the endoscopic classification for unitary groups [Mok, Kaletha–Minguez–Shin–White].

Special cycles on X

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- For any $y \in V$ with $(y, y) > 0$. Its orthogonal complement $V_y \subseteq V$ is standard indefinite of rank $m - 1$. The embedding $H_y = \mathrm{U}(V_y) \hookrightarrow H = \mathrm{U}(V)$ defines a Shimura subvariety of codimension 1

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- For any $\mathbf{x} = (x_1, \ldots, x_n) \in V(\mathbb{A}_f)^n$ with $T(\mathbf{x}) = ((x_i, x_j)) \in \mathrm{Herm}_n(F_0)_{>0}$, define the special cycle (of codimension n)

$$Z(\mathbf{x}) = Z(x_1) \cap \cdots \cap Z(x_n) \to X.$$

Geometric Siegel–Weil formula

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- More generally, for $\varphi \in \mathscr{S}(V(\mathbb{A}_f)^n)^K$ and $T \in \mathrm{Herm}_n(F_0)_{>0}$, define the weighted special cycle

$$Z(T,\varphi) = \sum_{\substack{\mathbf{x} \in K \backslash V(\mathbb{A}_f)^n \\ T(\mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x}) \in \mathrm{CH}^n(X)_{\mathbb{C}}.$$

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- With extra care, one can also define $Z(T,\varphi) \in \mathrm{CH}^m(X)_{\mathbb{C}}$ for any $T \in \mathrm{Herm}_n(F_0)_{\geq 0}$.
- Define Kudla's arithmetic theta series

$$Z(\tau,\varphi) = \sum_{T \in \mathrm{Herm}_n(F)_{\geq 0}} Z(T,\varphi) \cdot q^T,$$

$$\tau \in \mathcal{H}_n = \{\mathsf{x} + i\mathsf{y} : \mathsf{x} \in \mathrm{Herm}_n(F_{0,\infty}),\ \mathsf{y} \in \mathrm{Herm}_n(F_{0,\infty})_{>0}\}$$

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- Take $n = m-1$, then $Z(T,\varphi)$ has dimension 0.
- Its degree $\deg Z(T,\varphi)$ = geometric intersection number of n special divisors.

Theorem (Kudla, 1990s, Geometric Siegel–Weil formula)

Take $n = m-1$. Assume that X is compact. Then
$$\sum_{T\in\mathrm{Herm}_n(F_0)_{\geq 0}} \deg Z(T,\varphi)\cdot q^T \doteq E(\tau,1/2,\varphi).$$

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- For suitable levels K, one can construct a regular integral model $\mathcal{X}$ of (a PEL variant) of X, and also integral models of special cycles [Kudla–Rapoport].
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- Take $n = m$ so that $\mathcal{Z}(T, \varphi)$ ($T \in \mathrm{Herm}_n(F_0)_{>0}$) has "expected dimension" 0.
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- The nonsingular Fourier coefficient decomposes as

$$E_T'(\tau, 0, \varphi) = \sum_v E_{T,v}'(\tau, 0, \varphi).$$

Arithmetic Siegel–Weil formula

- For suitable levels K, one can construct a regular integral model $\mathcal{X}$ of (a PEL variant) of X, and also integral models of special cycles [Kudla–Rapoport].
- $Z(x) \to X$ extends to a special divisor $\mathcal{Z}(x) \to \mathcal{X}$.
- $\mathcal{X}$ is an arithmetic variety of dimension m.
- Take $n = m$ so that $\mathcal{Z}(T, \varphi)$ ($T \in \mathrm{Herm}_n(F_0)_{>0}$) has "expected dimension" 0.
- Its arithmetic degree encodes arithmetic intersection numbers at all places

$$\widehat{\deg}\, \mathcal{Z}(T, \varphi) \quad \text{"} = \text{"} \quad \sum_v \mathrm{Int}_{T,v}(\varphi).$$

- Kudla envisioned arithmetic Siegel–Weil formula

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- At $v \mid \infty$, proved by [Liu 2011] and [Garcia–Sankaran 2018] independently.
- At $v \nmid \infty$, the identity is the content of the Kudla–Rapoport conjecture.

Kudla–Rapoport Conjecture

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- $\Lambda_v \subseteq V_v$ a self-dual lattice with respect to the hermitian form.
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- Take $n = m$, $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_n \in \mathscr{S}(V(\mathbb{A}_f)^n)^K$ such that $\varphi_{i,v} = \mathbf{1}_{\Lambda_v}$.
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- $T \in \mathrm{Herm}_n(F_0)_{>0}$ with diagonal entries $t_1, \ldots, t_n$.
- Define arithmetic intersection number at v

$$\mathrm{Int}_{T,v}(\varphi) := \chi(\mathcal{Z}(T,\varphi)_v, \mathcal{O}_{\mathcal{Z}(t_1,\varphi_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} \cdots \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{\mathcal{Z}(t_n,\varphi_n)}) \cdot \log q_v$$

Theorem ([L.-Zhang 2019], Kudla–Rapoport Conjecture)

Assume that v is unramified in F. Take $n = m$. Then for any $T \in \mathrm{Herm}_n(F_0)_{>0}$,

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Here

- $\Phi_v \in \mathscr{S}(\mathbb{V}_v^n)$ is an explicit function independent of T, and
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- The conceptual recipe of the correction term (i.e. the choice of Φ_v) was conjectured by [He–Shi–Yang 2021], who also proved the special case $n = 2, 3$. The proof of the theorem is new even for $n = 2, 3$.

Applications to L-functions (via the doubling method)

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  central order of vanishing of $L_p(\pi)$ is 1 $\Longrightarrow \mathrm{rank}\,\mathrm{H}^1_f(F, V_\pi) \geq 1$.
- The Kudla-Rapoport conjecture is a key local ingredient in all these applications.

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Reduce to Local Kudla-Rapoport conjecture

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Next: define $\mathrm{Int}(L)$ and $\partial\mathrm{Den}(L)$.

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- More precisely, $\mathcal{N}$ is the formal scheme over $\mathrm{Spf}\, O_{\breve{F}}$ representing the functor

$$S \mapsto \mathcal{N}(S) = \{(X, \iota, \lambda, \mathcal{F}, \rho)\}/\mathrm{isom}.$$

  - (X, ι, λ): hermitian O_F-module of signature $(1, n-1)$ over S,
  - $\mathcal{F}$: local direct summand of $\mathrm{Lie}\, X$ of rank 1 as an $\mathcal{O}_S$-module such that O_F acts on $\mathcal{F}$ (resp. $\mathrm{Lie}\, X/\mathcal{F}$) through the embedding $O_F \to O_{\breve{F}}$ (resp. the conjugate embedding),
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- $\mathcal{N}$ provides a p-adic uniformization of $\widehat{\mathcal{X}}_{/\mathcal{X}^{\mathrm{ss}}_{\bar{k}}}$ at a ramified place.
- Two choices of the local hermitian space V (up to isometry) in local Shimura data:
    - n even: two non-isomorphic $\mathrm{U}(V)$, giving rise to two non-isomorphic $\mathcal{N}$,
    - n odd: two isomorphic $\mathrm{U}(V)$, giving rise to only one $\mathcal{N}$.

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- $n = 2$: $\chi(V) = -1$, $\mathcal{M} \simeq \widehat{\Omega}_{F_0} \times_{\mathrm{Spf}\, O_{F_0}} \mathrm{Spf}\, O_{\breve{F}}$, $\dim \mathcal{M}^{\mathrm{red}} = 1$.

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- There is another integral model $\mathcal{M}$ (Pappas model), which is flat but not regular. The Krämer model $\mathcal{N}$ provides a resolution of singularities
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with fiber of each singular point isomorphic to $\mathbb{P}^{n-1}{}_{/\bar{k}}$ ("exceptional divisor").
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- $n = 2$: $\chi(V) = +1$, $\mathcal{M} \simeq \mathrm{Spf}\, O_{\breve{F}}[[x, y]]/(xy - \pi^2)$, $\mathcal{M}^{\mathrm{red}} = \{\mathrm{pt}\}$,
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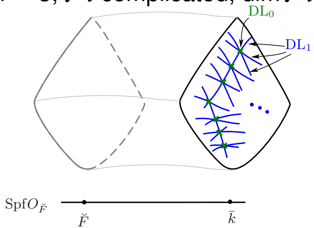

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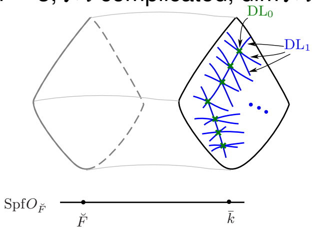

$\mathrm{DL}_0 = \{\mathrm{pt}\}$, a single point.

$\mathrm{DL}_1 \simeq \mathbb{P}^1$.

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- $\mathbb{V}$ can be identified with the space of special quasi-homomorphisms:

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- $L = \langle x_1, \ldots, x_n \rangle \subseteq \mathbb{V}$: O_F-lattice of rank n. Define the special cycle

$$\mathcal{Z}(L) := \mathcal{Z}(x_1) \cap \cdots \cap \mathcal{Z}(x_n) \subseteq \mathcal{N}.$$

- Define the arithmetic intersection number

$$\mathrm{Int}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}) \in \mathbb{Z}.$$

- $\mathrm{Int}(L)$ depends only on L [Howard]; it is nonzero only when L is integral, i.e. $L \subseteq L^\sharp$.

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- $\mathrm{Herm}_{L,M}$: the O_{F_0}-scheme of hermitian O_F-module homomorphisms from L to M.
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$$\mathrm{Den}(M, L) := \lim_{N \to +\infty} \frac{|\mathrm{Herm}_{L,M}(O_{F_0}/\pi^{2N})|}{q^{N \cdot d_{L,M}}},$$

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- Since $\chi(M) \neq \chi(L)$, we have $\mathrm{Den}(I_n, L) = 0$ and consider the derivative

$$\mathrm{Den}'(I_n,L) := -2 \cdot \frac{\mathrm{d}}{\mathrm{d}X}\bigg|_{X=1} \mathrm{Den}(I_n,L,X).$$

- Define the (normalized) derived local density

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Let Λ_t be a vertex lattice of type $t > 0$. Then $L = \Lambda_t^\sharp$ satisfies $L \not\subseteq L^\sharp$, while $L \subseteq L^\vee$, so:

$$\mathrm{Int}(\Lambda_t^\sharp) = 0, \text{ while } \mathrm{Den}'(\Lambda_t^\sharp) \neq 0.$$

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Then for any $T \in \mathrm{Herm}_n(F_0)_{>0}$,

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$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

Here

$$\partial\mathrm{Den}(L) := \mathrm{Den}'(L) + \sum_{j=1}^{t_{\max}/2} c_{2j} \cdot \mathrm{Den}_{2j}(L) \in \mathbb{Z}.$$

Theorem ([He–L.–Shi–Yang 2022], KR for Krämer models, precise version)

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$$\Phi_v := \sum_{j=1}^{t_{\max}/2} c_{2j} \cdot \mathbf{1}_{(\Lambda_{2j}^{\sharp})^n} \cdot \frac{\mathrm{vol}(\mathrm{U}(I_n))}{\mathrm{vol}(\mathrm{U}(\Lambda_{2j}^{\sharp}))} \cdot \log q_v \in \mathscr{S}(\mathbb{V}_v^n).$$

Then for any $T \in \mathrm{Herm}_n(F_0)_{>0}$,

$$\mathrm{Int}_{T,v}(\varphi) q^T \doteq E'_{T,v}(\tau, 0, \varphi) + E_T(\tau, 0, \varphi^v \otimes \Phi_v).$$

Remark

Also prove a closed formula for c_{2j} in terms of quadratic spaces over finite fields:

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Remarks on the main theorem

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It can be simplified depending on n and $\chi(\mathbb{V})$, e.g. when n is odd:

$$c_{2j} = \frac{(-1)^{n+j}}{q^{j(n-j-1)}(q^j+1)}.$$

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This theorem involves proving a lot of cancellation of terms. The cancellation is easier when L is “very integral”, harder when L is “slightly integral”, and hardest when L is “slightly non-integral”. The modification assumption exactly kicks in to simplify the hardest case.

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$$\mathrm{Int}_{L^\flat} = \partial\mathrm{Den}_{L^\flat}.$$

- $\mathrm{Int}_{L^\flat}$ is hard to compute due to improper intersection.
- $\partial\mathrm{Den}_{L^\flat}$ has a (complicated) lattice-theoretic formula.

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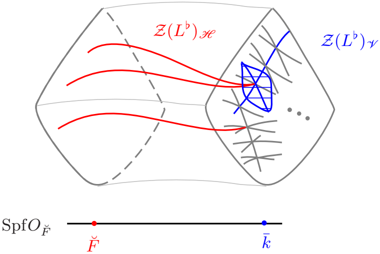

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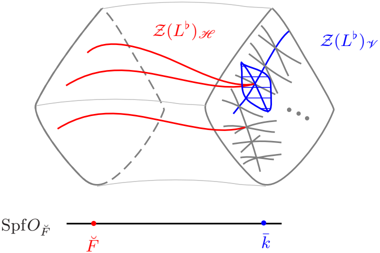

$$\mathrm{Int}_{L^\flat,\mathscr{H}}(x) := \chi(\mathcal{N}, \mathcal{Z}(L^\flat)_{\mathscr{H}} \cap^{\mathbb{L}} \mathcal{Z}(x)), \quad \mathrm{Int}_{L^\flat,\mathscr{V}}(x) := \mathrm{Int}_{L^\flat}(x) - \mathrm{Int}_{L^\flat,\mathscr{H}}(x).$$

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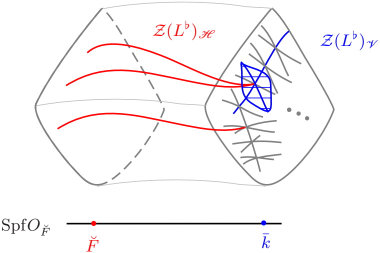

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The horizontal part $\mathcal{Z}(L^\flat)_{\mathscr{H}}$ can be understood in terms of Gross’ quasi-canonical lifting, and allows us to match

$$\mathrm{Int}_{L^\flat,\mathscr{H}} = \partial\mathrm{Den}_{L^\flat,\mathscr{H}}.$$

Thus it remains to prove the vertical identity

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Theorem (Key theorem)

(1) $\widehat{\mathrm{Int}^\perp_{L^\flat,\mathscr{V}}}$ is supported on $\mathbb{W}^{\geq -1} := \{x \in \mathbb{W} : \mathrm{val}_{F_0}(x,x) \geq -1\}$.

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Remark

In the unramified case, $\widehat{\mathrm{Int}_{L^\flat,\mathcal{V}}} = -\mathrm{Int}_{L^\flat,\mathcal{V}}$. This stronger invariance is not true in the Krämer case and (1) can be viewed as a weaker replacement.

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$$r(n^2) = ?, \qquad \text{for } n = 3 \cdot 13 \cdot 2023$$

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$$\begin{aligned}
(3 \cdot 13 \cdot 2023)^2 &= (\pm 78897)^2 + 0^2 &&= 0^2 + (\pm 78897)^2 \\
&= (\pm 78540)^2 + (\pm 7497)^2 &&= (\pm 7497)^2 + (\pm 78540)^2 \\
&= (\pm 77385)^2 + (\pm 15372)^2 &&= (\pm 15372)^2 + (\pm 77385)^2 \\
&= (\pm 72828)^2 + (\pm 30345)^2 &&= (\pm 30345)^2 + (\pm 72828)^2 \\
&= (\pm 69615)^2 + (\pm 37128)^2 &&= (\pm 37128)^2 + (\pm 69615)^2 \\
&= (\pm 65772)^2 + (\pm 43575)^2 &&= (\pm 43575)^2 + (\pm 65772)^2 \\
&= (\pm 65520)^2 + (\pm 43953)^2 &&= (\pm 43953)^2 + (\pm 65520)^2 \\
&= (\pm 61047)^2 + (\pm 49980)^2 &&= (\pm 49980)^2 + (\pm 61047)^2
\end{aligned}$$

Happy $r((3 \cdot 13 \cdot 2023)^2)$ -th Birthday
to Shou-Wu!