Exceptional theta functions

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Harmonic theta functions: Example

Define

$$S = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

- Let $V = \mathbf{R}^8$, $L = \mathbf{Z}^8$, and $(x, y) = \frac{1}{2}x^t S y$.
- S is positive definite, and $(\, , \,)$ is integral on L.
- This is the E_8 root lattice
- Set $p(v) = (e_1 + ie_2, v)^8$, $\Theta_{L,p}(z) = \sum_{v \in L} p(v) e^{2\pi i (v,v) z}$.
- $\Theta_{L,p}(z) = 1920\Delta(z) = 1920 \sum_{n \geq 1} \tau(n) e^{2\pi i n z}$

Theta function of a lattice

- Suppose V is a finite-dimensional real vector space with a positive-definite inner product $(\ ,\ ) : V \otimes V \to \mathbf{R}$
- Suppose $L \subseteq V$ is a lattice, such that $(x,x) \in \mathbf{Z}$, $(x,y) \in \frac{1}{2}\mathbf{Z}$ for all $x, y \in L$.

The theta function of a lattice

Define

$$\Theta_L(z) = \sum_{v \in L} e^{2\pi i (v,v) z}.$$

This is a modular form of weight $\dim(V)/2$.

Example

Suppose

- $V = \mathbf{R}^N$,
- $L = \mathbf{Z}^N$,
- $(\, , \,)$ the usual inner product on V.

Then

Standard lattice

$$\Theta_L(z) = \sum_{m \geq 0} r_N(m) q^m$$

where

$$r_N(m) = \#\{(x_1, \ldots, x_N) \in \mathbf{Z}^N : x_1^2 + \cdots + x_N^2 = m\}$$

is the number of ways of writing m as the sum of N squares.

- $\Theta_L(z)$ is a modular form of weight $N/2$ and level $\Gamma_1(4)$.

Harmonic theta functions

1. Let $V, L, (\, , \,)$ be as above
2. Suppose $w \in V \otimes \mathbf{C}$ is isotropic, i.e. $(w, w) = 0$. Define H_n to be the polynomials $V \to \mathbf{C}$ of degree n spanned by the $v \mapsto (w, v)^n$, w isotropic. These are called the harmonic polynomials. (This is a finite-dimensional irreducible representation of $\mathrm{SO}(V)$).

Harmonic theta functions

For $p \in H_n$, set

$$\Theta_{L,p}(z) = \sum_{v \in L} p(v) e^{2\pi i (v,v) z}.$$

This is a modular form of weight $\dim(V)/2 + n$. If $n > 0$ it is a cusp form.

Modern viewpoint

1. Let V be a positive definite rational quadratic space
2. Assume for simplicity that $\dim(V)$ is even
3. Using the Weil representation, can make (many) two-variable theta functions $\Theta(g, h)$, $g \in \mathrm{SL}_2(\mathbf{A})$, $h \in \mathrm{SO}(V)(\mathbf{A})$, which are automorphic forms in each variable
4. Given an automorphic form α on $\mathrm{SO}(V)$, one defines the theta lift of α,
$$\Theta(\alpha)(g) = \int_{[SO(V)]} \Theta(g, h)\alpha(h)\, dh.$$
This is an automorphic form on SL_2.
5. Because V is positive-definite, $SO(V)(\mathbf{Q})\backslash \mathrm{SO}(V)(\mathbf{A})$ is compact. Consequently, automorphic forms α on $\mathrm{SO}(V)$ can be described in terms of finite-dimensional representations of $\mathrm{SO}(V)(\mathbf{R})$ and combinatorial data.

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- Suppose (ρ, V_ρ) is a finite-dimensional representation of $\mathrm{GL}_n(\mathbf{C})$.
- Let $\mathcal{H}_n = \mathrm{Sp}_{2n}(\mathbf{R})/U(n)$ be the symmetric space for $\mathrm{Sp}_{2n}(\mathbf{R})$.
- One realizes $\mathcal{H}_n$ as

$$\mathcal{H}_n = \{Z \in M_n(\mathbf{C}) : Z^t = Z, Im(Z) > 0\}.$$

- $\mathrm{Sp}_{2n}(\mathbf{R})$ acts on $\mathcal{H}_n$ by linear fractional transformations: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{R})$ and $Z \in \mathcal{H}_n$ then

$$\gamma \cdot Z = (aZ + b)(cZ + d)^{-1}.$$

Siegel modular forms of weight ρ

A level one Siegel modular form of weight ρ is a holomormphic function $F : \mathcal{H}_n \to V_\rho$ satisfying $F(\gamma Z) = \rho(cZ + d)F(z)$ for all γ in $\mathrm{Sp}_{2n}(\mathbf{Z})$.

Fourier expansion

Siegel modular forms have a classical Fourier expansion:

- Let $S(\mathbf{Z}^n)^\vee$ denote the half-integral $n \times n$ matrices.
- $T \in S(\mathbf{Z}^n)^\vee$ if T is symmetric, with integer diagonal entries, and off-diagonal entries in $\frac{1}{2}\mathbf{Z}$.

Fourier expansion

If F is a level one Siegel modular form on Sp_{2n} of weight ρ, then

$$F(Z) = \sum_{T \in S(\mathbf{Z}^n)^\vee : T \geq 0} a_F(T) e^{2\pi i \,\mathrm{tr}(TZ)}$$

with $a_F(T) \in V_\rho$ called the **Fourier coefficients** of F.

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Two forms of G_2

Split G_2

Let G_2^s denote the split algebraic group of type G_2 over $\mathbf{Q}$

There is also a form of G_2 that is compact at the archimedean place:

Anisotropic G_2

There is a form G_2^c of G_2 over $\mathbf{Q}$ that is split at all finite places and such that $G_2^c(\mathbf{R})$ is compact.

The first part of the talk will be about G_2^c. The second part of the talk will be about G_2^s.

Algebraic modular forms I

- Suppose $\pi = \pi_f \otimes \pi_\infty$ is an automorphic representation of $G_2^c(\mathbf{A})$.
- Then π_∞ is an irreducible representation of $G_2^c(\mathbf{R})$ and thus is finite-dimensional.
- Let W be (the space of) this finite-dimensional representation.
- Automorphic forms φ in π can be described combinatorially in terms of vectors in W.

A finite group $G_2^c(\mathbf{Z})$

Set $G_2^c(\mathbf{Z}) := G_2^c(\mathbf{Q}) \cap G_2^c(\widehat{\mathbf{Z}})$. This is a finite group of order 12096.

Algebraic modular forms II

- Suppose W is a finite-dimensional irreducible representation of $G_2^c(\mathbf{R})$ over $\mathbf{C}$.
- Let $\mathcal{A}(G_2^c; W)$ be the space of level-one automorphic forms on G_2^c with coefficients in W.

I.e., $\varphi \in \mathcal{A}(G_2^c; W)$ if

$$\varphi : G_2^c(\mathbf{A}) \to W$$

is an automorphic form satisfying

1. $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in G_2^c(\mathbf{R})$ and $g \in G_2^c(\mathbf{A})$
2. $\varphi(gk_f) = \varphi(g)$ for all $k_f \in G_2^c(\widehat{\mathbf{Z}})$.

Lemma 1 (Well-known)

The map $\varphi \mapsto \varphi(1)$ defines a linear isomorphism

$$\mathcal{A}(G_2^c, W) \to W^{G_2^c(\mathbf{Z})}.$$

Langlands functoriality

- The dual group of G_2^c is $G_2(\mathbf{C})$.
- $G_2(\mathbf{C})$ has a 7-dimensional (standard) representation, that lands in $\mathrm{SO}_7(\mathbf{C})$.
- Recall that $\mathrm{SO}_7(\mathbf{C})$ is the dual group of Sp_6.
- Langlands functoriality predicts a lift from automorphic representations of G_2^c to automorphic representations of Sp_6.

This conjectural lift was studied by Gross-Savin:

Gross-Savin

- There is a dual pair $G_2^c \times \mathrm{Sp}_6 \subseteq E_{7,3}$.
- The group $E_{7,3}$ has a minimal representation (H. Kim), which can be used as a Θ-kernel to (sometimes) understand this conjectural lift
- Elements of $\mathcal{A}(G_2^c; W)$ should lift to vector-valued Siegel modular forms of a prescribed weight.

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Theta lifting

- Recall that there is a minimal representation Π_{min} on $E_{7,3}$ which can be used as a Θ-kernel to lift automorphic forms from G_2^c to Siegel modular forms on Sp_6
- If π is an automorphic representation of $G_2^c(\mathbf{A})$, let $\Theta(\pi)$ be its lift to Sp_6 using the various Θ_ϕ's for $\phi \in \Pi_{min}$

The following proposition follows easily from work of Gan-Savin, Magaard-Savin, Gross-Savin:

Proposition 2

Suppose π on G_2^c is unramified at every finite place, and $\pi_\infty = W$. Suppose moreover that $\Theta(\pi)$ is nonzero. Then

1. $\Theta(\pi)$ is generated by a level one Siegel modular form F_π;
2. The weight of F_π is explicitly determined by W;
3. F_π is a Hecke eigenform, with Satake parameters $c_p \in G_2(\mathbf{C}) \subseteq \mathrm{SO}_7(\mathbf{C})$ for all p.

A π as on the previous slide corresponds to a vector

$$\varphi_\pi \in A(G_2^c; W), \text{ or equivalently a } \alpha_\pi \in W^{G_2^c(\mathbf{Z})}.$$

Theorem 3

Let the notation be as above. Then the Fourier expansion of F_π can be given completely explicitly in terms of α_π.

Corollary 4

There is an algorithm to determine if a cuspidal Siegel modular form F of most weights is a theta lift from G_2^c.

Let $\rho_1 = [(12, 8, 8)]$; $\rho_2 = [(14, 10, 8)]$.

It is known (Chenevier-Taibi) that there are unique level one Siegel modular cusp forms F_1, F_2 of these weights, up to scalar multiple.

Corollary 5

F_1 and F_2 are theta lifts from G_2^c. In particular, all their Satake parameters are in $G_2(\mathbf{C})$.

Contents

Modular forms on G_2

- $G_2^s(\mathbf{R})$: a noncompact simple Lie group of dimension 14
- $K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$ is a maximal compact subgroup of $G_2^s(\mathbf{R})$
- $\mathbf{V}_\ell := Sym^{2\ell}(\mathbf{C}^2) \boxtimes \mathbf{1}$, $\ell \geq 1$ integer, a representation of K.

Definition (Gross-Wallach, Gan-Gross-Savin)

Suppose $\Gamma \subseteq G_2^s(\mathbf{R})$ a congruence subgroup. A modular form on G_2^s of weight ℓ and level Γ is a smooth, moderate growth function $\varphi : \Gamma \backslash G_2^s(\mathbf{R}) \to \mathbf{V}_\ell$ satisfying

1. $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K$ and
2. $D_\ell \varphi \equiv 0$ for a certain special linear differential operator D_ℓ.

Modular forms on G_2^s have a Fourier expansion

Definition

A real binary cubic form $f(u,v) = au^3 + bu^2v + cuv^2 + dv^3$, $a,b,c,d \in \mathbf{R}$, is said to be positive semi-definite, $f \geq 0$, if $f(z,1)$ is never 0 on the upper half-plane $\mathfrak{h}$. Equivalently, $f \geq 0$ if f factors into three linear factors over $\mathbf{R}$.

Theorem 6 (P.)

Fix $\ell \geq 1$. There exist explicit functions $W_f : G_2^s(\mathbf{R}) \rightarrow \mathbf{V}_\ell$, if $f \geq 0$, satisfying

- $W_f(gk) = k^{-1} \cdot W_f(g)$ *for all* $k \in K$
- $D_\ell W_f(g) \equiv 0$.

If φ a modular form of weight ℓ and level Γ (sufficiently large), then

$$\varphi(g) \text{``}=\text{''} \sum_{f \geq 0, f \text{ integral}} a_\varphi(f) W_f(g)$$

for some $a_\varphi(f) \in \mathbf{C}$.

Some remarks

Remark

The existence of the Fourier coefficients (without the explicit functions W_f), at least for non-degenerate f, was given by Gan-Gross-Savin, crucially using a result of Wallach.

Remark

The $a_\varphi(f)$ are defined in a very transcendental way. There is no *a priori* reason that they might be connected to arithmetic.

Conjecture (P.)

There exists a basis of modular forms of weight ℓ such that all the Fourier coefficients of elements of this basis are algebraic numbers.

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Modular forms on G_2^s

- There is a group F_4^c, of type F_4, that is split at every finite place and compact at the archimedean place
- Similar to $\mathrm{Sp}_6 \times G_2^c \subseteq E_{7,3}$, there is $G_2^s \times F_4^c \subseteq E_{8,4}$.
- The minimal representation (Gan) on $E_{8,4}$ can be used to lift (algebraic) modular forms on F_4^c to (quaternionic) modular forms on G_2^s.

Let V_1 be the irreducible representation of $F_4^c(\mathbf{R})$ of dimensional 273, and V_m the irrep with highest weight m times the highest weight of V_1.

Theorem 7

Suppose $m \geq 1$. There is a lattice $\Lambda_m \subseteq V_m$ so that if $\alpha \in \Lambda_m$, then the theta lift $\Theta(\alpha)$ is a cuspidal quaternionic modular form on G_2^s of weight $4+m$ with completely explicit Fourier expansion. Its Fourier coefficients are all integers.

Proof sketch

$\mathrm{Sp}_6 \times G_2^c \subseteq E_{7,3}$

1. Apply differential operators to Kim's modular form on $E_{7,3}$
2. Do this enough times until you are in the "right" K-type of the minimal representation
3. Integrate out the φ, and all "bad" terms vanish

$G_s^2 \times F_4^c \subseteq E_{8,4}$

1. Start with Gan's Theta function on $E_{8,4}$ which generates the minimal representation on this group
2. I calculated its Fourier expansion a few years ago
3. Apply differential operators to it until you are in the right K-type (roughly)
4. Previous step is now substantially harder
5. Integrate out the φ, and miraculously get a simple formula

Applications of a result of Dalal

Theorem 8 (Dalal)

There is an explicit formula for the dimension of the cuspidal quaternionic level one modular forms on G_2^s of weights $\ell \geq 3$.

- Using Dalal's formula, in weight less than 12, I have checked on my laptop that every cuspidal QMF is a lift from F_4^c, and thus these modular forms have a basis with integral Fourier coefficients
- The dimension of the space of such QMFs is 9
- If every level one cuspidal QMF is a lift from F_4^c, then one can tabulate a database of the Fourier expansion of level one cuspidal modular forms on G_2^s

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Explicit formula Sp_6 I

- Let Θ be the octonions with positive definite norm form, V_7 its trace 0 elements, and $H_3(\Theta)$ the exceptional Jordan algebra of 3×3 Hermitian matrices over Θ
- Suppose $T \in H_3(\Theta)$ is rank one
- By taking the trace 0 part of the off diagonal elements T, we obtain an element $v_1 \otimes x_1 + v_2 \otimes x_2 + v_3 \otimes x_3$ of $V_3 \otimes V_7$
- There is a natural map

$$(V_3 \otimes V_7)^{\otimes(k_1+2k_2)} \to S^{k_1}(V_3) \otimes V_7^{\otimes k_1} \otimes S^{k_2}(\wedge^2 V_3) \otimes (\wedge^2 V_7)^{\otimes k_2}.$$

- Denote by $P_{k_1,k_2}(T)$ the image of $T^{\otimes(k_1+2k_2)}$ under this map.

$$P_{k_1,k_2}(T) = (v_1 x_1 + v_2 x_2 + v_3 x_3)^{k_1}(w_1(x_2 \wedge x_3) + w_2(x_3 \wedge x_1) + w_3(x_1 \wedge x_2))^{k_2}$$

where $w_j = v_{j+1} \wedge v_{j+2}$ and indices are taken modulo 3.

- The irrep W is the highest weight submodule of $V_7^{\otimes k_1} \otimes (\wedge^2 V_7)^{\otimes k_2}$ for unique non-negative integers k_1, k_2.
- If $\beta \in W$, denote by $\{P_{k_1,k_2}(T), \beta\}$ the natural pairing, valued in $S^{k_1}(V_3) \otimes S^{k_2}(\wedge^2 V_3)$.
- Finally, for T "integral" in $H_3(\Theta)$, set $a(T)$ the Fourier coefficient of T in Kim's modular form on $E_{7,3}$, so that $a(T) = 240\sigma_3(d_T)$ where d_T measures how divisible is T.

Theorem 9

Suppose $\beta \in W$ is such that $\alpha_\pi = \frac{1}{|G_2^c(\mathbf{Z})|} \sum_{\gamma \in G_2^c(\mathbf{Z})} \gamma\beta$. Then

$$F_\pi(Z) = \sum_{T \text{ rank } 1} a(T)\{P_{k_1,k_2}(T), \beta\} e^{2\pi i \operatorname{tr}(TZ)}.$$

Algebraic modular forms on F_4^c

Proposition 10 (Gross)

The double coset
$$F_4^c(\mathbf{Q})\backslash F_4^c(\mathbf{A}_f)/F_4^c(\widehat{\mathbf{Z}})$$
has size two.

- Let V be a finite-dimensional representation of $F_4^c(\mathbf{R})$
- As a consequence of the proposition, level one algebraic modular forms for F_4^c can be described as elements of $V^{\Gamma_I} \oplus V^{\Gamma_E}$ for certain finite groups Γ_I and Γ_E

The representation V_m

Set $J = H_3(\Theta)$, and J^0 the subspace with 0 trace. There is an exact sequence
$$0 \to V_1 \to \wedge^2 J^0 \to \mathfrak{f}_4 \to 0.$$
Thus one can find $V_m \subseteq (\wedge^2 J)^{\otimes m}$.

Explicit formula for G_2^s

- Set $W_J = \mathbf{Z} \oplus J \oplus J \oplus \mathbf{Z}$. This is an integral model for the 56-dimensional representation of E_7.
- There is a notion of "rank one" elements of W_J, which are those elements in the E_7 orbit of a highest weight vector
- For $w \in W_J$, define $a_\Theta(w) = \sigma_4(d_w)$ if w is rank one and 0 otherwise. Here d_w is the largest integer such that $w \in d_w W_J$.
- For $w = (a, b, c, d) \in W_J$, set $P_m(w) = (b \wedge c)^{\otimes m} \in (\wedge^2 J)^{\otimes m}$. Note that $\langle P_m(w), \beta \rangle \in \mathbf{C}$.
- Denote $pr_I(w) = au^3 + \mathrm{tr}(b)u^2v + \mathrm{tr}(c)uv^2 + dv^3$

Theorem 11

Suppose $m > 1$ and $\beta \in V_m$. Then

$$\sum_{w \in W_J} a_\Theta(w) \langle P_m(w), \beta \rangle W_{pr_I(w)}(g)$$

is the Fourier expansion of a level one cuspidal QMF on G_2^s of weight $4 + m$.

Happy Birthday Shou-Wu!