

Bigness of Admissible Canonical Bundle

(1) Uniform Bogomolov

curve: Smooth, proj, geom. int.

Thm (Ulmo, Bogomolov Conj.)

$C/\bar{\mathbb{Q}}$ curve, genus $g > 1$

$\alpha \in \text{Pic}^1(C)$

$i_\alpha: C \rightarrow \mathcal{J}, x \mapsto x - \alpha$

\exists constant $\varepsilon > 0$ s.t.

$$\# \left\{ x \in C(\bar{\mathbb{Q}}) \mid \begin{array}{c} \hat{h}(i_\alpha(x)) < \varepsilon \\ \parallel \\ \hat{h}(x - \alpha) \end{array} \right\} < \infty.$$

Thm (Dimitrov-Gao-Habegger, Kühne)

Fix $g > 1$.

$\exists c_1, c_2$ dep. only on g $\begin{pmatrix} c_1, c_2 \\ \text{positive} \end{pmatrix}$ s.t.

$\forall X/\bar{\mathbb{Q}}$ curve, genus g

$\forall x_0 \in \underline{X(\bar{\mathbb{Q}})}$

$$\# \left\{ x \in X(\bar{\mathbb{Q}}) \mid \hat{h}(x-x_0) < \underbrace{c_1 \max\{h_{\text{Fal}}(x), 1\}}_{c_1} \right\} < \underline{c_2}$$

Here $h_{\text{Fal}}(X) = h_{\text{Fal}}(J(X))$

Thm (Uniform Mordell)

$\forall X/\mathbb{C}$, genus $g > 1$.

$\forall T \subseteq J(\mathbb{C})$ fin. rk (for some $X \hookrightarrow J$)

we have

$$\# (X(\mathbb{C}) \cap T) \leq c(g)^{\text{rk } T + 1}$$

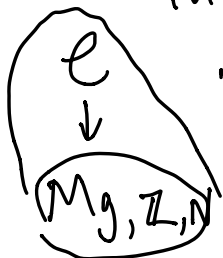
Remark: This is implied by

①. (DhH, K) $\# \{ \text{small pts} \} < ?$

②. (Vojta's proof) $\# \{ \text{large pts} \} < ?$

Thm (Y.) Fix $g > 1$

Then $\exists c_1 > 0, c_2 > 0$ dep. only on g , s.t.



$\forall K$ either number field

or fun. field of 1 var. over k

$\forall X/\bar{K}$ curve, genus g

$\forall \alpha \in \text{Pic}^1(X)$,

assuming (X, α) non-isotrivial in fun. fld. case

$$\# \left\{ x \in X(\bar{K}) \mid \hat{h}(x-\alpha) < c_1 \left(\max\{h_{\text{Fal}}(c), 1\} + \hat{h}((2g-2)\alpha - \omega_{X/\bar{K}}) \right) \right\} < c_2.$$

(extra term)
(uniform Mumford)

Rmk: Loozer-Silverman-Wilms proved the result for function fields independently.

(a). no extra term

(b). explicit constants c_1, c_2 .

(admissible pairing on single curve X)

Ideas:

(1). (DGH, K) non-degeneracy (model theory)
ht ineq., equidistribution

(2). $(Y.)$ adelic line bundles by Y.-Zhang
bigness of admissible canonical sheaf

(2) Adm. canonical line bundle

Recall K num. field

X/K quasi-proj

An adelic line bundle is $(L, \{\|\cdot\|_v\}_{v \in M_K})$

①. L lb on X

②. $\|\cdot\|_v$ metric of L on $X_{K_v}^{\text{an}}$
(or $X(\bar{K}_v)$)

③. the metric is limit of metrics induced by
proj. models $(\mathcal{X}, \mathcal{L})$ of (X, L) over \mathcal{O}_K
($X \hookrightarrow \mathcal{X}_K$ open) by boundary top.

Recall: case X/K curve, $g > 1$.

There there is an admissible adelic metric

$\|\cdot\|_v$ on $\omega_{X/K}$ ($v \in M_K$)

①. $v \neq \infty$, Arakelov metric

②. $v = \infty$, Zhang metric

\Rightarrow adelic l.b. $\bar{\omega}_{X/K}$ on X/K .

($\bar{\omega}_{X/K} > 0$ \Leftrightarrow Bogomolov conj.)
(Zhang)

(This path is finished by Zhang, de Jong, Cinkir)

Rmk. my approach is family version of this path.

admissible: $C \xrightarrow{i} J$, $x \mapsto (2g-2)x - \omega_{X/K}$
 $\bar{\omega}_{X/K} \cong i^* \bar{\theta}$ $\leftarrow \bar{\theta}$ (canonical metric
 $[2]^* \bar{\theta} = 4 \bar{\theta}$)
 $\omega_{X/K}^{\otimes e} \cong i^* \theta$

Family version

S/K quasi-proj var

$\pi: X \rightarrow S$ smooth, proj, fibers are curves of genus $g > 1$

(e.g. $S = M_{g,N}$)

There $\omega_{X/S}$ extends to a canonical adelic line bundle $\bar{\omega}_{X/S}$ on X .

(putting metrics on $X_{K_v}^{\text{an}}$, $v \in M_K$)

Idea: patch $\{\bar{\omega}_{X_y/y} \mid y \in S \text{ closed}\}$
to form $\omega_{X/S}$.

Key formula (J relative Jacobian)

$$\begin{array}{ccc}
 X & \xrightarrow{i} & J \\
 \pi \downarrow & \swarrow & \\
 S & &
 \end{array}
 \quad
 \begin{array}{l}
 \kappa \mapsto (2g-1)\kappa - \omega_{X/S} \\
 \Theta \text{ lb on } J \text{ sym, rel. ample} \\
 \bar{\Theta} \text{ canon. metric: } [2J]^* \bar{\Theta} = 4\bar{\Theta}. \\
 \text{(adelic lb)}
 \end{array}$$

Denote $\bar{L} = i^* \bar{\Theta}$ on X .

$$\textcircled{1}. \quad \bar{L} = 4g(g-1) \bar{\omega}_{X/S} - \pi^* \langle \bar{\omega}_{X/S}, \bar{\omega}_{X/S} \rangle$$

$$\textcircled{2}. \quad \langle \bar{L}, \bar{L} \rangle = 16g(g-1)^3 \langle \bar{\omega}, \bar{\omega} \rangle$$

Here $\langle \bar{L}, \bar{L} \rangle$ Deligne pairing (lb on S)

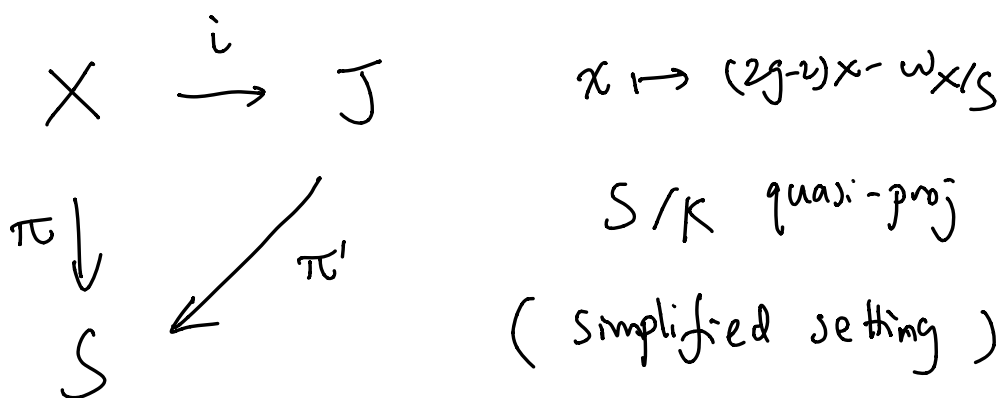
Consequence: $\bar{\omega}_{X/S}$ nef, as \bar{L} nef.
(by $\textcircled{1}, \textcircled{2}$)

Thm Assume S max variation
 ($S \rightarrow M_g$ generically finite)

Then $\bar{\omega}_{X/S}$ is big.

(top. self. int. > 0)

(3). Proof of Unif. Bog.



$$i^* \bar{\Theta} = \bar{L} \text{ on } X$$

want $h_{\bar{L}}(x) = h_{\bar{\Theta}}(i(x))$ "big" for most $x \in X(\bar{K})$

If \bar{L} is big, this is obtained by the ht ineq.

For any Weil ht $h_M: S(\bar{K}) \rightarrow \mathbb{R}$,

$\exists \epsilon > 0$ s.t.

$\{ x \in X(\bar{K}) \mid h_{\bar{L}}(x) < \varepsilon h_M(\pi(x)) \}$
 not zar. dense in X .

Problem: \bar{L} is never big for the case

Solution: $\langle \bar{L}, \bar{L} \rangle = c \cdot \langle \bar{\omega}, \bar{\omega} \rangle$
 is big on S .
 (conseq. of bigness of $\bar{\omega}$)

$\Rightarrow \bar{L}$ potentially big, i.e.

$\forall m > \dim S$,

an $X^m/S = X \times_S \dots \times_S X$,

the adelic lb

$$L^{\boxtimes m} = p_1^* \bar{L} \otimes \dots \otimes p_m^* \bar{L}$$

is big.

$$\left(\underbrace{\text{top. int. of } L^{\boxtimes m}} > \underbrace{\text{top. int. of } \langle \bar{L}, \bar{L} \rangle} \right)$$

Then apply ht ineq. for $L^{\boxtimes m}$ on X^m/S .

hts on X^m/S "big"

\Rightarrow ht on X is "big".