Bigness of Admissible Canonical Bundle

(1) Uniform Bogomolov

curve: smooth, proj, geom. int.

Thm (Ullmo, Bogomolov Conj.)

$C/\bar{\mathbb{Q}}$ curve, genus $g>1$

$\alpha \in \mathrm{Pic}^1_C(C)$

$i_\alpha : C \to J$, $\quad x \mapsto x-\alpha$

$\exists$ constant $\varepsilon > 0$ s.t.

$$\#\{ x \in C(\bar{\mathbb{Q}}) \mid \underbrace{\hat{h}(i_\alpha(x))}_{= \hat{h}(x-\alpha)} < \varepsilon \} < \infty.$$

Thm (Dimitrov-Gao-Habegger, Kühne)

Fix $g>1$.

$\exists\, c_1, c_2$ dep. only on g (c_1, c_2 positive) s.t.

$\forall\, X/\bar{\mathbb{Q}}$ curve, genus g

$\forall\, x_0 \in X(\bar{\mathbb{Q}})$

$$\#\left\{ x \in X(\bar{\mathbb{Q}}) \,\middle|\, \hat{h}(x - x_0) < c_1 \max\{ h_{Fal}(X), 1\} \right\} < c_2$$

c_1 $\qquad\qquad$ c_2

Here $h_{Fal}(X) = h_{Fal}(J(X))$

Thm (Uniform Mordell)

$\forall\ X/\mathbb{C}$, genus $g > 1$.

$\forall\ \Gamma \subseteq J(\mathbb{C})$ fin. rk (for some $X \hookrightarrow J$)

we have

$$\#\left(X(\mathbb{C}) \cap \Gamma\right) \leq c(g)^{\operatorname{rk}\Gamma + 1}.$$

Rmk: This is implied by

①. (DGH, K) $\#\{\text{small pts}\} < ?$

②. (Vojta's proof) $\#\{\text{large pts}\} < ?$

Thm (Y.) Fix $g > 1$

Then $\exists\ c_1 > 0$, $c_2 > 0$ dep. only on g, s.t.

$\mathcal{C} \downarrow \mathcal{M}_{g,\mathbb{Z},N}$

$\forall\ K$ either number field

or fun. field of 1 var. over k

$\forall$ $X/\bar{K}$ curve, genus g

$\forall$ $\alpha \in \mathrm{Pic}^1(X)$,

assuming (X, α) non-isotrivial in fun. fld. case

$$\#\left\{x \in X(\bar{K}) \,\middle|\, \hat{h}(x-\alpha) < c_1\left(\max\{h_{\mathrm{Fal}}(C), 1\} + \hat{h}((2g-2)\alpha - \omega_{X/K})\right)\right\}$$

$< c_2$.

(extra term)

(Uniform Mumford)

Rmk: Looper-Silverman-Wilms proved the result for function fields independently.

(a). no extra term

(b). explicit constants c_1, c_2.

(admissible pairing on single curve X)

Ideas:

(1). (DGH, K) non-degeneracy (model theory)

ht ineq., equidistribution

(2). (Y.) adelic line bundles by Y.-Zhang

bigness of admissible canonical sheaf

(2). Adm, canonical line bundle

Recall K num. field

X/K quasi-proj

An adelic line bundle is $(L, \{\|\cdot\|_v\}_{v\in M_K})$

(1). L lb on X

(2). $\|\cdot\|_v$ metric of L on $X^{an}_{K_v}$

(or $X(\bar{K}_v)$)

(3). the metric is limit of metrics induced by proj. models $(\mathcal{X}, \mathcal{L})$ of (X, L) over O_K

($X \hookrightarrow \mathcal{X}_K$ open) by boundary top.

Recall: case $X/_K$ curve, $g>1$.

Then there is an admissible adelic metric

$\|\cdot\|_v$ on $\omega_{X/K}$ ($v\in M_K$)

(1). $v|\infty$, Arakelov metric

(2). $v\nmid\infty$, Zhang metric

$\Rightarrow$ adelic l.b. $\bar{\omega}_{X/K}$ on X/K.

($\tilde{\bar{\omega}}_{X/K} > 0 \iff$ Bogo conj.)
(Zhang)

(This path is finished by Zhang, de Jong, Cinkir)

Rmk, my approach is family version of this path.

admissible: $C \xrightarrow{i} J$, $x \mapsto (2g-2)x - \omega_{X/K}$

$\bar{\omega}_{X/K}^{\otimes e} \cong i^*\bar{\Theta} \leftarrow \bar{\Theta}$ (canonical metric $[2]^*\bar{\Theta} = 4\bar{\Theta}$)

$\omega_{X/K}^{\otimes e} \cong i^*\Theta$

Family version

S/K quasi-proj var

$\pi: X \to S$ smooth, proj, fibers are curves of genus $g>1$

(e.g. $S = M_{g,N}$)

Then $\omega_{X/S}$ extends to a canonical adelic line bundle $\bar{\omega}_{X/S}$ on X.

(putting metrics on $X_{K_v}^{an}$, $v \in M_K$)

Idea: patch $\{\bar{\omega}_{X_y/y} \mid y \in S \text{ closed}\}$
to form $\omega_{X/S}$.

Key formula (J relative Jacobian)

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\pi"'] & J \arrow[dl] \\ S & \end{tikzcd}$$

$x \mapsto (2g-2)x - \omega_{X/S}$

Θ lb on J sym, rel. ample

$\bar{\Theta}$ canon. metric: $[2]^*\bar{\Theta} = 4\bar{\Theta}$.
(adelic lb)

Denote $\bar{L} = i^*\bar{\Theta}$ on X.

①. $\bar{L} = 4g(g-1)\,\bar{\omega}_{X/S} - \pi^*\langle \bar{\omega}_{X/S}, \bar{\omega}_{X/S}\rangle$

②. $\langle \bar{L}, \bar{L}\rangle = 16g(g-1)^3 \langle \bar{\omega}, \bar{\omega}\rangle$

Here $\langle \bar{L}, \bar{L}\rangle$ Deligne pairing (lb on S)

Consequence: $\bar{\omega}_{X/S}$ nef, as $\bar{L}$ nef.
(by ①, ②)

Thm Assume S max variation

($S \to M_g$ generically finite)

Then $\overline{\omega}_{X/S}$ is big.

(top. self. int. > 0)

(3). Proof of Unif. Bog.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\pi"'] & J \arrow[dl, "\pi'"] \\ S & \end{tikzcd}$$

$x \mapsto (2g-2)x - \omega_{X/S}$

S/K quasi-proj

(simplified setting)

$i^* \overline{\Theta} = \overline{L}$ on X

want $h_{\overline{L}}(x) = h_{\overline{\Theta}}(i(x))$ "big" for most $x \in X(\overline{K})$

If $\overline{L}$ is big, this is obtained by the ht ineq.:

For any Weil ht $h_M : S(\overline{K}) \to \mathbb{R}$,

$\exists\ \varepsilon > 0$ s.t.

$$\{ x \in X(\overline{K}) \mid h_{\overline{L}}(x) < \varepsilon\, h_M(\pi(x)) \}$$

not zar. dense in X.

<u>Problem</u>: $\overline{L}$ is never big for the case

<u>Solution</u>: $\langle \overline{L}, \overline{L} \rangle = c \cdot \langle \overline{\omega}, \overline{\omega} \rangle$

is big on S.

(conseq. of bigness of $\overline{\omega}$)

$\Rightarrow$ $\overline{L}$ <u>potentially big</u>, i.e.

$\forall\, m > \dim S$,

on $X^m_{/S} = X \times_S \times \cdots \times_S X$,

the adelic lb

$$L^{\boxtimes m} = p_1^* \overline{L} \otimes \cdots \otimes p_m^* \overline{L}$$

is big.

$$\Big(\underbrace{\text{top. int. of } L^{\boxtimes m}} > \underbrace{\text{top. int. of } \langle \bar{L}, \bar{L} \rangle} \Big).$$

Then apply ht ineq. for $L^{\boxtimes m}$ on $X^m_{/S}$.

hts on $X^m_{/S}$ "big"

$\Rightarrow$ ht on X is "big".