Joint unlikely almost intersections on ordinary Siegel spaces

Congling Qiu

Yale University

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Outline

- Motivations
- Unlikely almost intersections
- Ax–Lindemann principle
- Perfectoid approach to unlikely almost intersections

Motivations

André–Oort vs André–Pink

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- Progress by Pink, Edixhoven–Yafaev, Orr, Richard–Yafaev.

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- No direct way.
- But for any variety over a valued field, $\mathbb{R}$-valued distance from points to a subvariety is defined. E.g., local heights.

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Theorem (Q)

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- Want an analog for Hecke orbit (though the exact analog may fail).

Sample theorem

Theorem (Q)

Let $S = \prod S_i$ be a product of modular curves with good reduction at p, and $V \subset S$ a curve but not geodesic. Let $O = \prod O_i$ where $O_i \subset S_i$ is CM or a Hecke orbit. Then $V_\epsilon \cap O$ has a finite set of reduction at p, if the p-adic distance ϵ is small enough.

Unlikely almost intersections

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- Noot proved "weakly special $\Rightarrow$ weakly linear".
  Moonen proved the converse, assuming algebraicity and that the translated formal subtori are torsion translates.

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- Bad news: analog of Mordell–Lang for formal groups fails (by Serban).
- My answer: not sure.

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- But both $\mathcal{V}(\overline{F}^\circ) \cap O_\epsilon$ and $\mathcal{V}_\epsilon \cap O$ have reductions in $\mathcal{V}_k$. (The latter one has larger reduction.)

Conjecture (Unlikely almost intersections)

If $\mathcal{V}$ is reduced and flat over F°, and the reduction of $\mathcal{V}_\epsilon \cap O$ is Zariski dense in $\mathcal{V}_k$ for all $\epsilon > 0$, then $\mathcal{V}$ is weakly linear.

Naive joint unlikely intersections

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Conjecture (André–Oort + André–Pink)

Let S_1, S_2 be Shimura varieties. Let $V \subset S_1 \times S_2$ be a closed subvariety. If $V \cap (\mathrm{CM}_1 \times O_2)$ is Zariski dense in V. Then V is weakly special. Here $\mathrm{CM}_1 \subset S_1$ is the set of CM points and $O_2 \subset S_2$ a Hecke orbit.

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- Special case of the Zilber–Pink conjecture.

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Let O be the Hecke saturation of a weakly special subset of S. Let $V \subset S$ be a closed subvariety. If $V \cap O$ is Zariski dense in V. Then V is weakly special.

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- Partial progress

Theorem (Q)

Let $O \subset \mathcal{S}(\overline{F}^{\circ})$ be the saturation under prime-to-p Hecke action and (forward and backward) Frobenius action of a weakly special subset of $\mathcal{S}(\overline{F}^{\circ})$. Assume that the reduction of $\mathcal{V}_\epsilon \cap O$ is Zariski dense in $\mathcal{V}_k$ for all $\epsilon > 0$. Then there is a nonempty open subscheme of $\mathcal{V}_k$ such that for every x of its k-points, $\mathcal{V}_x$ contains a translated formal subtorus of $\mathcal{S}_x$.

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- The Frobenius endomorphism on $\mathcal{S}_k$ admits the "canonical lifting" to $\mathcal{S}$. It is a p-primary Hecke action.
- In fact, can allow "partial Frobenii". E.g., if $\mathcal{S}$ is replaced by a product of modular curves, O is a product CM's and Hecke orbits.

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Conjecture (Weakly linear Ax–Lindemann)

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Theorem

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- Proof of (2). A global toric action on an Igusa scheme.

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- On $\mathcal{I}_y$, the action is just group multiplication.

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- Let $\mathcal{M}/F^\circ$ be the deformation space of (polarized) ordinary p-divisible groups of height $2g$, i.e., $\mathcal{M} := \{(G, G_k \cong \widehat{\mathbb{G}}_m^g \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g\}$.
- Induces polarized $0 \to \widehat{\mathbb{G}}_m^g \to G \to (\mathbb{Q}_p/\mathbb{Z}_p)^g \to 0$.
- $\mathcal{M} \cong \widehat{\mathbb{G}}_m^{g(g+1)/2}$, group law is Baer sum of extensions.
- Serre–Tate: $\mathcal{M} \cong \mathcal{S}_x$. But not canonical.
- Igusa scheme: a pro-finite-étale cover of $\mathcal{S}$,

$$\mathcal{I} := \{A \in \mathcal{S}, \text{ and polarized } 0 \to \widehat{\mathbb{G}}_m^g \to A[p^\infty] \to (\mathbb{Q}_p/\mathbb{Z}_p)^g \to 0\}.$$

- Then $\mathcal{M} \cong \mathcal{I}_y$ canonically by the choice of y.
- $\mathcal{M} \curvearrowright \mathcal{I}$ by Baer sum of extensions and so on (Liu, S. Zhang, W. Zhang).
- On $\mathcal{I}_y$, the action is just group multiplication.
- Use $\mathcal{M} \curvearrowright \mathcal{I}$ to extend local properties, e.g., linearity.

Perfectoid approach to U.A.I.

Set up

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- Canonical lifting

$$\mathcal{I}(F^\circ) \cap \mathrm{CM} \xleftarrow{\mathrm{can}} \mathcal{I}\,(k)$$

$$\text{identity of } \mathcal{I}_y \hookleftarrow y.$$

Perfectoid Igusa spaces

- Igusa scheme in [Caraiani–Scholze, 2017]

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- Tilting bijection

$$\rho : \mathcal{I}^{\mathrm{perf}}(C^\circ) \cong \mathcal{I}_k^{\mathrm{perf}}\left(C^{\flat\circ}\right)$$

$$P \mapsto P^\flat$$

if $P/p = P^\flat/t$ (under $C^\circ/p \cong C^{\flat\circ}/t$).

Perfectoid Igusa spaces

- An enhancement of can. Precisely,

$$
\begin{array}{ccc}
\mathcal{I}^{\mathrm{perf}}(C^\circ) & \xleftarrow[\cong]{\rho^{-1}} & \mathcal{I}_k^{\mathrm{perf}}\left(C^{\flat\circ}\right) \\
\downarrow & & \updownarrow \cong \\
\mathcal{I}(C^\circ) & & \mathcal{I}_k\left(C^{\flat\circ}\right) \\
\uparrow & & \uparrow \\
\mathcal{I}(F^\circ) & \xleftarrow[\mathrm{can}]{} & \mathcal{I}_k(k)
\end{array}
$$

commutes, where we recall

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- And the diagram respects Frobenii.

Scholze's approximation lemma, Xie

- $$\begin{array}{ccc}
\mathcal{I}^{\mathrm{perf}}(C^\circ) & \xleftarrow[\cong]{\rho^{-1}} & \mathcal{I}_k^{\mathrm{perf}}\left(C^{\flat\circ}\right) \\
\Big\downarrow \pi & & \Big\updownarrow \cong \\
\mathcal{W}(C^\circ) \subset \mathcal{I}(C^\circ) & & \mathcal{I}_k\left(C^{\flat\circ}\right) \supset \mathcal{W}_k\left(C^{\flat\circ}\right) \\
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- $$\begin{array}{ccc} \mathcal{I}^{\mathrm{perf}}(C^\circ) & \xleftarrow[\cong]{\rho^{-1}} & \mathcal{I}_k^{\mathrm{perf}}\left(C^{\flat\circ}\right) \\ \downarrow \pi & & \updownarrow \cong \\ \mathcal{W}(C^\circ) \subset \mathcal{I}(C^\circ) & & \mathcal{I}_k\left(C^{\flat\circ}\right) \supset \mathcal{W}_k\left(C^{\flat\circ}\right) \\ \uparrow & & \uparrow \\ \mathcal{I}(F^\circ) & \xleftarrow[\mathrm{can}]{} & \mathcal{I}_k(k) \end{array}$$
- Want to lift the Frobenius stability of $\mathcal{W}_k\left(C^{\flat\circ}\right)$ to $\mathcal{W}$.

Lemma

Let $\Lambda_n \subset \mathcal{I}_k(k)$ such that $\mathcal{W}_k \subset \Lambda_n^{\mathrm{Zar}}$. If $\mathrm{can}(\Lambda_n) \subset \mathcal{W}_{1/n}$ for all n, then

$$\pi\left(\rho^{-1}\left(\mathcal{W}_k\left(C^{\flat\circ}\right)\right)\right) \subset \mathcal{W}(C^\circ).$$

The End

Thank you