Regularity for fluid-structure interactions and its relation to uniqueness

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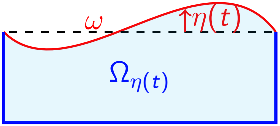
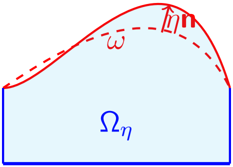
Examples & Setup

Consider an elastic plate interacting with a fluid



or an elastic shell interacting with a fluid



The solid deforms in **Lagrangian coordinates** w.r.t. a reference state. $\eta : [0, T] \times \omega \to \mathbb{R}$.
The fluid by **Eulerian coordinates** on the *time-changing geometry* Ω_η via its velocity $\mathbf{v} : [0, T] \times \Omega_\eta \to \mathbb{R}^d$ and pressure $p : [0, T] \times \Omega_\eta \to \mathbb{R}$.

The PDEs for thin solids

A perfect elastic solid is driven by its *elastic energy* $\mathcal{E}$

$$h\rho_s\partial_t^2\eta(t) = -\mathcal{E}'(\eta(t)) + g \text{ in } [0, T] \times \omega.$$

a visco-elastic solid additionally by its dissipation potential $\mathcal{R}$

$$h\rho_s\partial_t^2\eta(t) = -\mathcal{E}'(\eta(t)) - D_2\mathcal{R}(\eta, \partial_t\eta) + g \text{ in } [0, T] \times \omega.$$

We have the following dichotomies.

Visco-elasticity	Elasticity
Plate	Shell
Linear	Non-linear
Normal displacement	Free displacement

Linear plate: $\mathcal{E}'(\eta) = \alpha\Delta^2\eta - \beta\Delta\eta$, $D_2\mathcal{R}(\eta, \partial_t\eta) = \alpha_0\Delta^2\partial_t\eta - \beta_0\Delta\partial_t\eta$.
Linear shell: $\mathcal{E}'(\eta) = \alpha\Delta^2\eta - \mathcal{L}\eta$, $D_2\mathcal{R}(\eta, \partial_t\eta) = \alpha_0\Delta^2\partial_t\eta - \mathcal{L}_0\partial_t\eta$.
Non-linear Koiter energy: **G** represents area and **R** curvature change

$$\mathcal{E}_K(\eta) = \frac{h}{4}\int_\omega \mathcal{A}\mathbf{G}(\eta(t,.)) : \mathbf{G}(\eta(t,.))dS + \frac{h^3}{48}\int_\omega \mathcal{A}\mathbf{R}(\eta(t,.)) : \mathbf{R}(\eta(t,.))dS$$

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PDEs for fluid-strcuture interaction

The solid defines the fluid domain

$$h\rho_s\partial_t^2\eta(t) = -\mathcal{E}'(\eta(t)) + g + g_f \text{ in } [0,T]\times\omega.$$

The movement of the Fluid is governed by Navier Stokes equation:

$$\begin{aligned} \operatorname{div}(\mathbf{v}) &= 0, && \text{in } [0,T]\times\Omega_\eta,\\ \rho_f(\partial_t(\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes\mathbf{v})) &= \mu\Delta\mathbf{v} - \nabla p + \mathbf{f} && \text{in } [0,T]\times\Omega_\eta, \end{aligned}$$

Physical quantities: h thickness of the shell, ρ_s solid density, ρ_f fluid density μ fluid viscosity.
Coupling 1: Boundary values, $\mathbf{v}(\eta) = \partial_t\eta\mathbf{n}$.
Coupling 2: Equilibrium of forces
$g_f(t,y) = -|\mathbf{n}(\eta(t,y))|(Ip - \nu\nabla_{\mathrm{sym}}\mathbf{v})(t,\eta(t,y))\mathbf{n}(\eta(t,y))\cdot\mathbf{n}(y)$

Weak formulation–coupled momentum equation

For all

$$(\xi(t), \psi(t)) \in H^2(\omega) \times H^1(\Omega_\eta) \text{ such that } \xi(t,y) = \psi(t, \eta(t,y))$$

the following is satisfied[1]

$$\frac{d}{dt}\left(\int_{\Omega_\eta} \mathbf{v} \cdot \psi \mathrm{dx} + \int_\omega \partial_t \eta \xi \, dS \right) - \int_{\Omega_\eta} \mathbf{v} \cdot \partial_t \psi + \mathbf{v} \otimes \mathbf{v} : \nabla \psi \mathrm{dx} - \int_\omega \partial_t \eta \, \partial_t \xi \, dS$$
$$+ \int_{\Omega_\eta} \left(\nabla_{\mathrm{sym}} \mathbf{v} : \nabla_{\mathrm{sym}} \psi - p \, \mathrm{div} \, \psi \right) \mathrm{dx} + \langle \mathcal{E}'(\eta), \xi \rangle = \langle \mathbf{f}, \psi \rangle_{\Omega_\eta} + \langle g, \xi \rangle_\omega$$

Take $(\partial_t \eta, \mathbf{v})$ as test function, then (by Korn's inequality)

$$\frac{d}{dt}\left(\frac{\|\mathbf{v}(t)\|^2_{L^2(\Omega_\eta)}}{2} + \frac{\|\partial_t \eta(t)\|^2_{L^2(\omega)}}{2} + \mathcal{E}(\eta(t)) \right) + \int_{\Omega_\eta} |\nabla \mathbf{v}|^2 \, dx \leq C(\mathbf{f}, g)$$

[1] We set $h = \rho_f = \rho_s = \mu = 1$.

Existence results–Cauchy problem

Visco-elasticity	Elasticity
Plate	Shell
Linear	Non-linear
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Weak solutions to Navier-Stokes coupled with rather general thin solids are available.

- **Plates:** Chambolle, Desjardins, Esteban, Grandmont, '05; Grandmont, '08
- **Visco-elastic linear shells**: Muha, Canic '13
- **Linear elastic shells:** Lengeler, Ruzicka '14
- **Nonlinear elastic shells:** Muha, Sch, '22
- **Nonlinear elastic plates with free displacement 2D:** Kampschulte, Sch, Sperone '23

Ladyzhenskaya-Prodi-Serrin condition

For 3D Navier-Stokes only conditional uniqueness is known.
If the *Ladyzhenskaya-Prodi-Serrin condition*

$$\mathbf{u} \in L^r(0, T; L^s(\Omega)), \qquad \frac{2}{r} + \frac{3}{s} = 1, \qquad 2 \leq r < \infty.$$

is satisfied, solutions are

- Smooth (as was shown by Ladyzhenskaya)
- Unique *in the class of all weak solutions* (as was shown by Prodi and Serrin). This relates to weak-strong uniqueness.

The borderline case $s = 3$, $r = \infty$ is of particular interest: It suffices for uniqueness *if a smooth right hand side is considered* (Kozono, Sohr '96, Escauriaza-Seregin-Sverak '03), but non-uniqueness is known in case a singular forcing is considered (Albritton-Brue-Colombo '22).

A first weak-strong uniqueness result

Theorem (Sch, Sorczinski '22 (for plates))

Let (v_1, p_1, η_1) *and* (v_2, p_2, η_2) *be weak solutions and assume for some* $s > 3$ *that* $v_2 \in L^2(0, T; W^{1,s}(\Omega_{\eta_2}))$ *and* $\partial_t v_2 \in L^2(0, T; W^{-1,2}(\Omega_{\eta_2}))$. *If* $v_1(0) = v_2(0)$, $\eta_1(0) = \eta_2(0)$, $\partial_t \eta_1(0) = \partial_t \eta_2(0)$ *then* $(v_1, p_1, \eta_1) = (v_2, p_2, \eta_2)$.

Previous works on uniqueness

1. Weak-strong uniqueness *rigid body motions*: (Glass, Sueur '19), (Chemetov, Necasova, Muha, '19), (Kreml, Necasova, Piaseck '20), (Necasova, Muha, Radosevic '21), *time-periodic* (Galdi '22) .
2. Global existence of smooth for visco-elastic fluids in 2-D (including $-\Delta \partial_t \eta_t$): (Grandmont, Hillariet '16).
3. Local existence of smooth solutions (2D): (Coutand, Shkoller '06,'07), (Boulakia '07), (Grandmont-Hillariet '19).
4. Global existence with small data (bulk) (Chueshov, Lasiecka, Webster '13).
5. Weak-strong uniqueness for *compressible fluids interacting with (heat-conducting) plates* (Trifunovic '23)

Strategy for uniqueness

Strategy: Subtract the two systems and use the difference of solutions as test-function.

First problem: The two geometries are different.

Solution: Use a change of variables. $\overline{v_2}(t,x,y) = v_2\big(t,x,\frac{\eta_1(t,x)}{\eta_2(t,x)}y\big)$

Second problem: This function is not divergence free.

Strategy 1: Use a Bogovskij operator: $\mathrm{div}(\mathcal{B}f) = f$ in Ω_η, $f = 0$ on $\partial\Omega_\eta$.

Problem with Bogovskij: How to estimate $\partial_t \mathcal{B} f$?

Strategy 2: Direct approach use *Piola transform* , which conserves the divergence.

Third problem: One cannot test.

Solution part 1: Test E_2 (strong) with $(\eta_2 - \eta_1, v_2 - \overline{v}_1)$, E_1 (weak) with $(\eta_2, \mathcal{P}_\eta v_2)$ and add the energy inequality for (η_1, v_1).

Collected terms are formally well defined.

Solution part 2: The time-derivatives do not exist. E.g.: $\langle \Delta\eta_1, \Delta\partial_t\eta_2 \rangle$ is not defined.

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Distributional time derivative

Lemma (Sch, Sorczinski '22 (for plates))

Let (v, η) be a weak solution.
If $v \in L^2(0, T; W^{1,s}(\Omega_\eta(t)))$ for $s \geq 2$ then

$$\partial_t v + [\nabla v]v \in L^2(0, T; (W^{1,q}_{0,\mathrm{div}}(\Omega_\eta(t)))^*)$$

for any $q \in (2, \infty)$ if $s = 2$ and $q = 2$ if $s > 2$.

$$\int_0^T \langle \partial_t v + [\nabla v]v, \varphi \rangle_{\Omega_\eta} \, dt = -\int_0^T \int_{\Omega_\eta(t)} \nabla v \cdot \nabla \varphi \, dx \, dt.$$

Moreover, the pair $(\partial_t v + [\nabla v]v, \partial_t^2 \eta) \in L^2(0, T; \mathcal{W}^*)$ for

$$\mathcal{W} = \{(\varphi, b) \in W^{1,q}_{\mathrm{div}}(\Omega_\eta(t)) \times H^2(\omega) \, : \, \varphi(t, x, \eta(x)) = (0, b(t, x))^T\}$$

The proof strongly relies on (Muha, Sch 2022): $\eta \in L^2(H^s)$ for $s < \frac{1}{2}$

Ladyzhenskaya-Prodi-Serrin condition for shells

For shells the Piola-transform is not well defined–a new strategy is needed: Regularity implies uniqueness here.

Theorem (Breit, Mensah, Sch, Su 23' for shells)

Let $(\mathbf{v}, \eta)$ be a weak solution to Navier-Stokes coupled to

$$\partial_t^2 \eta - \Delta \partial_t \eta + \Delta^2 \eta = g_f .$$

Suppose that

$$\mathbf{v} \in L^r(I; L^s(\Omega_\eta)), \quad \frac{2}{r} + \frac{3}{s} \leq 1, \quad \eta \in L^\infty(I; C^1(\omega))$$

Then $(\mathbf{v}, \eta)$ is a strong solution.
Moreover, $(\mathbf{v}, \eta)$ is unique in the class of weak solutions satisfying the energy inequality with Lipschitz deformation.

Regularity check: $\eta \in H^2$ implies almost Lipschitz continuity.

Proof strategy

The proof contains three independent results (all new for shells).

1. **Local strong solutions.** The existence of a smooth solution for short times is constructed.
2. **The acceleration estimate.** As long as the Ladyzhenskaya-Prodi-Serrin condition is satisfied and the displacement of the shell stays C^1 in space, the solutions is a strong solution. Here the viscosity of the shell is essential!
3. **Weak-strong uniqueness.** Finally, it is shown that the constructed smooth solution is unique in the regime of weak solutions with bi-Lipschitz-in-space shell displacement.

Space regularity

The key to regularity in fluid-structure interaction is to improve the time-regularity, as the steady Stokes theory is well established:

Theorem (Breit '23)

Let $p \in (1,\infty)$, $s \geq 1+\frac{1}{p}$ and natural restrictions to ρ. Suppose that $\mathcal{O}$ is a $\mathbf{B}^{\theta}_{\varrho,p}$-domain for some $\theta > s - 1/p$ with locally small Lipschitz constant, RHS $\mathbf{f} \in W^{s-2,p}(\mathcal{O})$ and compatible boundary value $\mathbf{u}_\partial \in W^{s-1/p,p}(\partial\mathcal{O})$. Then there is a unique solution $(\mathbf{u},\pi)$ to the steady Stokes equation satisfying

$$\|\mathbf{u}\|_{W^{s,p}(\mathcal{O})} + \|\pi\|_{W^{s-1,p}(\mathcal{O})} \lesssim \|\mathbf{f}\|_{W^{s-2,p}(\mathcal{O})} + \|\mathbf{u}_\partial\|_{W^{s-1/p,p}(\partial\mathcal{O})}.$$

Time-regularity

We follow (Grandmont, Hillairet '16) and use the test-function $(\partial_t^2\eta, \partial_t \mathbf{v} + F_\eta(\partial_t\eta)\cdot\nabla\mathbf{v})$, where F_η is an extension operator into Ω_η.
Observe that $\partial_t^2\eta$ is not a good test function for hyperbolic equations!
However, for the visco-elastic solid testing with $\partial_t^2\eta$ implies

$$\int_\omega \left|\partial_t^2\eta\right|^2 + \partial_t \frac{\left|\nabla\partial_t\eta\right|^2}{2}\,dx = \int_\omega -g_f\partial_t^2\eta + \left|\Delta\partial_t\eta\right|^2\,dx.$$

Further testing with $-\Delta\partial_t\eta$ implies

$$\int_\omega \partial_t \frac{\left|\nabla\partial_t\eta\right|^2 + \left|\nabla\Delta\eta\right|^2}{2} + \left|\Delta\partial_t\eta\right|^2\,dx = -\int_\omega g_f\Delta\partial_t\eta\,dx,$$

this combination produces enough good terms on the left hand side to close the estimate.

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Bogovskij for Lipschitz domains

Paying the price of assuming Lipschitz deformations we get the following universal Bogovskij operator.

Theorem (Kampschulte, Sch, Sperone '23)

There is a universal Bogovskij operator, such that for all Ω_η with $\|\nabla\eta\|_\infty \le C_L$, $\|\eta\|_\infty \le L$ and $b \in C_0^\infty(\Omega \setminus S_L)$ with unit integral

$$\mathcal{B} : C_0^\infty(\Omega_\eta) \to C_0^\infty(\Omega_\eta; \mathbb{R}^n) \text{ with } \mathrm{div}\mathcal{B}f = f - b\int f\,\mathrm{d}x.$$

In addition $\|\mathcal{B}(f)\|_{W^{s+1,p}(\Omega_\eta;\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\Omega_\eta)}$ with C independent of η.

In particular $\partial_t\mathcal{B}(f\chi_{\Omega_\eta}) = \mathcal{B}(\partial_t f\chi_{\Omega_\eta})$ and $\mathcal{B}(\partial_t f\chi_{\Omega_\eta}) = 0$ on $\partial\Omega_\eta$.

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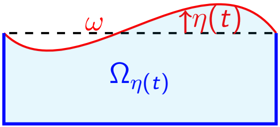
Strong solution for elastic plates

The time-regularity estimate strongly depends on the viscosity of the solid. If the solid is purely elastic a different strategy is needed. This is already true for short times (see M. Badra and T. Takahashi '19, '22)

Theorem (Sch, Su 23')

An elastic beam $\rho_s \partial_t^2 \eta + \alpha \partial_x^4 \eta - \beta \partial_x^2 \eta = g$ interacting with the 2D Navier-Stokes equation has a strong solution for arbitrary large times if no collision appears.

Proof idea: Take the time-derivative of the whole coupled system.



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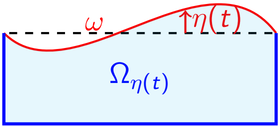

Centre of analysis and numerics for fluid-structure interactions at Charles University

https://fsi.karlin.mff.cuni.cz/

The Faculty of Mathematics and Physics of Charles University, Prague opens **two postdoc positions**within the **ERC-CZ Grant LL2105**, supported by the Ministry of Education, Youth and Sport of the Czech Republic: *"The interaction of fluids and solids", https://fsi.karlin.mff.cuni.cz/*

The **postdoc positions** are for **3 years** (1+2). The earliest possible start is January 2024 and should be filled by October 2024.

In case of interest please send your application until **10.12.2023** by email to **schwarz@karlin.mff.cuni.cz**. The application should be a single PDF file and include a CV, a research statement and the copy of the PhD diploma or if not available the master diploma. Two letters of recommendation should be sent separately. **We also have free PhD positions! In case of interest please contact us!**