A Mixed Variational Formulation for the Qualitative and Quantitaive Analysis of a Certain Compressible Flow – Incompressible Fluid PDE Interaction

George Avalos, University of Nebraska-Lincoln
Paula Egging, University of Nebraska-Lincoln

Workshop on Recent Progress in Deterministic and Stochastic Fluid-Structure Interaction

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Introduction

Partial differential equations (PDEs) can be used to model natural phenomena, including:

- Sound waves
- Heat dispersion
- Thermodynamics
- Fluid dynamics

And our present concern

- Ocean-atmosphere interaction, inspired by an internship project at Argonne National Lab.

Figure: free-images.com/display/ocean_water_sky_sea.html

Nebraska University of Lincoln

Introduction

Wellposedness and Numerical Results for a Fluid-Fluid Model

George Avalos and Paula Egging

Historically, semigroup generation in 3-D fluid-structure interaction models have been well-studied, including

- I. Chueshov, I. Ryzhkova, *A global attractor for a fluid-plate interaction model*, 2013.
- I. Chueshov, I. Lasiecka, J. T. Webster, *Flow-plate interactions: Well-posedness and long-time behavior*, 2014.
- L. Bociu, L. Castle, K. Martin, and D. Toundykov, *Optimal Control in a Free Boundary Fluid-Elasticity Interaction*, 2015.
- G. Avalos, P. G. Geredeli, J. T. Webster, *Semigroup Well-posedness of A Linearized, Compressible Fluid with An Elastic Boundary*, 2018.
- G. Avalos, P. G. Geredeli and B. Muha, *Rational Decay of A Multilayered Structure-Fluid PDE System*, 2022.

But the extension of similar techniques to fluid-fluid interaction has remained relatively untouched.

Our Model

Here, the geometry is

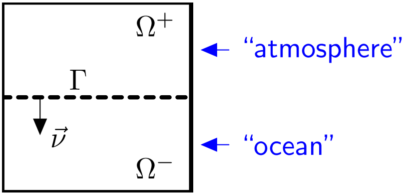


Figure: The fluid-fluid geometry.

The $\mathbf{u}^+$, $\mathbf{u}^-$ represent *velocity* of the fluid in Ω^+, Ω^-. The p^+, p^- represent *pressure* in Ω^+, Ω^-, respectively.

Additionally, $\mathbf{U}$ is a steady state solution to Navier-Stokes about which we linearize, and $\sigma(\mathbf{u}^+)$ is the *stress tensor* of $\mathbf{u}^+$.

Our Model

For variables $[\mathbf{u}^+, p^+, \mathbf{u}^-, p^-]$, consider the system:

$$\begin{cases} \mathbf{u}_t^+ + \mathbf{U}\cdot\nabla\mathbf{u}^+ - \operatorname{div}\sigma(\mathbf{u}^+) + \nabla p^+ = 0 & \text{on } \Omega^+ \times (0,T), \\ p_t^+ + \mathbf{U}\cdot\nabla p^+ + \operatorname{div}(\mathbf{u}^+) = 0 & \text{on } \Omega^+ \times (0,T), \\ \mathbf{u}^+ = 0 & \text{on } (\partial\Omega^+ \setminus \Gamma) \times (0,T), \end{cases}$$

(a compressible fluid evolving in time on Ω^+) (1)

$$\begin{cases} \mathbf{u}_t^- - \Delta\mathbf{u}^- + \nabla p^- = 0 & \text{on } \Omega^- \times (0,T), \\ \operatorname{div}(\mathbf{u}^-) = 0 & \text{on } \Omega^- \times (0,T), \\ \mathbf{u}^- = 0 & \text{on } (\partial\Omega^- \setminus \Gamma) \times (0,T), \end{cases}$$

(an incompressible fluid evolving in time on Ω^-) (2)

$$\begin{cases} \mathbf{u}^+ = \mathbf{u}^- & \text{on } \Gamma \times (0,T), \\ \sigma(\mathbf{u}^+)\vec{\nu} - p^+\vec{\nu} = \frac{\partial \mathbf{u}^-}{\partial \vec{\nu}} - p^-\vec{\nu} & \text{on } \Gamma \times (0,T), \\ \mathbf{u}^+(t=0) = \mathbf{u}_0^+; \quad \mathbf{u}^-(t=0) = \mathbf{u}_0^-. & \end{cases}$$

(boundary and initial conditions) (3)

Elimination of Pressure in Ω^-

We eliminate p^- by identifying it as the solution to the boundary value problem

$$\begin{cases} \Delta p^- = 0 & \text{on } \Omega^- \times (0,T), \\ p^- = \frac{\partial \mathbf{u}^-}{\partial \vec{\nu}} \cdot \vec{\nu} - [\sigma(\mathbf{u}^+)\vec{\nu}] \cdot \vec{\nu} + p^+ & \text{on } \Gamma \times (0,T), \\ \frac{\partial p^-}{\partial \vec{\nu}} = \Delta \mathbf{u}^- \cdot \vec{\nu} & \text{on } \partial\Omega^- \setminus \Gamma \times (0,T), \end{cases} \tag{4}$$

which is derived from (**??**) - (**??**).

Elimination of Pressure in Ω^-

Let the Dirichlet and Neumann maps, respectively, be given by

$$\mathbf{h} = D_s(\mathbf{g}) \iff \begin{cases} \Delta \mathbf{h} = \mathbf{0} & \text{on } \Omega^-, \\ \mathbf{h} = \mathbf{g} & \text{on } \Gamma, \\ \frac{\partial \mathbf{h}}{\partial \vec{\nu}} = \mathbf{0} & \text{on } \partial\Omega^- \setminus \Gamma, \end{cases}$$

and

$$\mathbf{h} = N_s(\mathbf{g}) \iff \begin{cases} \Delta \mathbf{h} = \mathbf{0} & \text{on } \Omega^-, \\ \mathbf{h} = \mathbf{0} & \text{on } \Gamma, \\ \frac{\partial \mathbf{h}}{\partial \vec{\nu}} = \mathbf{g} & \text{on } \partial\Omega^- \setminus \Gamma, \end{cases}$$

Elimination of Pressure in Ω^-

In consideration of the boundary conditions in (**??**),

$$p^-(t) = D_s\left(\frac{\partial \mathbf{u}^-(t)}{\partial \vec{\nu}}\cdot\vec{\nu} - [\sigma(\mathbf{u}^+(t))\vec{\nu}]\cdot\vec{\nu} + p^+(t)\right) + N_s(\Delta\mathbf{u}^-(t)\cdot\vec{\nu}) \ \in L^2(\Omega^-)$$

Then with

$$G_1\mathbf{u}^- = -\nabla\left(D_s\left(\frac{\partial \mathbf{u}^-}{\partial\vec{\nu}}\cdot\vec{\nu}\right) + N_s(\Delta\mathbf{u}^-\cdot\vec{\nu})\right),$$

$$G_2\mathbf{u}^+ = -\nabla\left(D_s([\sigma(\mathbf{u}^+)\vec{\nu}]\cdot\vec{\nu})\right); \quad G_3p^+ = -\nabla(D_s(p^+)),$$

we identify

$$\nabla p^- = -G_1\mathbf{u}^- - G_2\mathbf{u}^+ - G_3p^+ \text{ in } \Omega^-\times(0,T).$$

So we have ∇p^- in terms of $\mathbf{u}^+$, p^+, and $\mathbf{u}^-$.

Our Model

To determine the semigroup, consider the system again

$$\begin{cases} \mathbf{u}_t^+ + \mathbf{U}\cdot\nabla\mathbf{u}^+ - \operatorname{div}\sigma(\mathbf{u}^+) + \nabla p^+ = 0 & \text{on } \Omega^+ \times (0,T), \\ p_t^+ + \mathbf{U}\cdot\nabla p^+ + \operatorname{div}(\mathbf{u}^+) = 0 & \text{on } \Omega^+ \times (0,T), \\ \mathbf{u}^+ = 0 & \text{on } (\partial\Omega^+ \setminus \Gamma) \times (0,T), \end{cases}$$

$$\begin{cases} \mathbf{u}_t^- - \Delta\mathbf{u}^- + \nabla p^- = 0 & \text{on } \Omega^- \times (0,T), \\ \operatorname{div}(\mathbf{u}^-) = 0 & \text{on } \Omega^- \times (0,T), \\ \mathbf{u}^- = 0 & \text{on } (\partial\Omega^- \setminus \Gamma) \times (0,T), \end{cases}$$

$$\begin{cases} \mathbf{u}^+ = \mathbf{u}^- & \text{on } \Gamma \times (0,T), \\ \sigma(\mathbf{u}^+)\vec{\nu} - p^+\vec{\nu} = \frac{\partial \mathbf{u}^-}{\partial \vec{\nu}} - p^-\vec{\nu} & \text{on } \Gamma \times (0,T), \\ \mathbf{u}^+(t=0) = \mathbf{u}_0^+; \ \ \mathbf{u}^-(t=0) = \mathbf{u}_0^-. \end{cases}$$

Semigroup Generation

Keeping time derivatives on left and moving everything else to RHS, we have

$$\begin{cases} \mathbf{u}_t^+ = -\mathbf{U}\cdot\nabla\mathbf{u}^+ + \operatorname{div}\sigma(\mathbf{u}^+) - \nabla p^+ & \text{on } \Omega^+\times(0,T), \\ p_t^+ = -\operatorname{div}(\mathbf{u}^+) - \mathbf{U}\cdot\nabla p^+ & \text{on } \Omega^+\times(0,T), \\ \mathbf{u}_t^- = G_2\mathbf{u}^+ + G_3 p^+ + \Delta\mathbf{u}^- + G_1\mathbf{u}^- & \text{on } \Omega^-\times(0,T). \end{cases}$$

This is equivalent to the following system of equations

$$\frac{d}{dt}\begin{bmatrix}\mathbf{u}^+\\ p^+\\ \mathbf{u}^-\end{bmatrix} = \begin{bmatrix} -\mathbf{U}\cdot\nabla\mathbf{u}^+ + \operatorname{div}\sigma(\mathbf{u}^+) - \nabla p^+ \\ -\operatorname{div}(\mathbf{u}^+) - \mathbf{U}\cdot\nabla p^+ \\ G_2\mathbf{u}^+ + G_3p^+ + \Delta\mathbf{u}^- + G_1\mathbf{u}^- \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\mathbf{U}\cdot\nabla(\cdot) + \operatorname{div}\sigma(\cdot) & -\nabla(\cdot) & 0 \\ -\operatorname{div}(\cdot) & -\mathbf{U}\cdot\nabla(\cdot) & 0 \\ G_2 & G_3 & \Delta(\cdot) + G_1 \end{bmatrix}}_{\text{hopeful semigroup generator, } \mathcal{A}} \begin{bmatrix}\mathbf{u}^+\\ p^+\\ \mathbf{u}^-\end{bmatrix}. \quad (5)$$

Domain of Semigroup

We carefully choose the domain, $\mathcal{D}(\mathcal{A})$, to ensure the necessary regularity of solutions and that $\mathcal{A}$ is, indeed, a maximal dissipative generator.

Let the space of finite energy be

$$\mathcal{H} = \mathbf{L}^2(\Omega^+) \times L^2(\Omega^+) \times \{\mathbf{f} \in \mathbf{L}^2(\Omega^-) : \text{div}(\mathbf{f}) = 0 \text{ and } \mathbf{f} \cdot \vec{\nu}|_{\partial\Omega^- \setminus \Gamma} = 0\}.$$

A few key properties include

- $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$
- $\mathcal{D}(\mathcal{A}) \subset \mathbf{H}^1_{\partial\Omega^+ \setminus \Gamma}(\Omega^+) \times L^2(\Omega^+) \times \mathbf{H}^1_{\partial\Omega^- \setminus \Gamma}(\Omega^-)$
- $\mathbf{u}^+ = \mathbf{u}^-$ on Γ
- $[\mathbf{u}^+, p^+, \mathbf{u}^-] \in \mathcal{D}(\mathcal{A})$ if there exists a $p^- \in L^2(\Omega^-)$ such that $\nabla p^- = -G_1\mathbf{u}^- - G_2\mathbf{u}^+ - G_3 p^+$.

Main Theorem

Theorem (P.E., G. A., 2022)

(i) *The operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is maximal dissipative. Therefore, by the Lumer-Phillips Theorem, it generates a C_0-semigroup of contractions $\{e^{\mathcal{A}t}\}_{t\geq 0}$ on $\mathcal{H}$.*

(ii) *In particular, let $\lambda > 0$ and $[\mathbf{f}, g, \mathbf{h}] \in \mathcal{H}$ be given. (By part (i), there exists $[\mathbf{u}^+, p^+, \mathbf{u}^-] \in \mathcal{D}(\mathcal{A})$ which solves $(\lambda I - \mathcal{A})[\mathbf{u}^+, p^+, \mathbf{u}^-] = [\mathbf{f}, g, \mathbf{h}]$.)*
Then $\mathbf{u}^-$ and p^- can be characterized as the solution to a certain variational system, while $\mathbf{u}^+$ and p^+ can be characterized by

$$
\begin{aligned}
\mathbf{u}^+ &= \mu_\lambda(\mathbf{u}^-) + \tilde{\mu}([\mathbf{f}, g]^T) \\
p^+ &= q_\lambda(\mathbf{u}^-) + \tilde{q}([\mathbf{f}, g]^T),
\end{aligned}
$$

where $[\mu_\lambda, q_\lambda]$ and $[\tilde{\mu}, \tilde{q}]$ are (to be given) mappings.

Outline

The proof strategy for Part (i) is:

1. Show $\mathcal{A}$ is maximal dissipative.
2. Apply the classical Lumer-Phillips Theorem to obtain a C_0-semigroup of contractions, $\{e^{\mathcal{A}t}\}$.
3. This allows for solutions $[\mathbf{u}^+(t), p^+(t), \mathbf{u}^-(t)]$ of (**??**) - (**??**) to be obtained by applying $\{e^{\mathcal{A}t}\}$ to initial data $[\mathbf{u}_0^+, p^+(t=0), \mathbf{u}_0^-]$.

The characterizations of $\mathbf{u}^+$, p^+, $\mathbf{u}^-$, and p^- given in Part (ii) are obtained within the proof of Part (i).

Slight Caveat..

There is a slight caveat... $\mathcal{A}$, as defined, is *not* actually dissipative due to the non-zero $\mathbf{U}$.

However, the bounded perturbation

$$\hat{\mathcal{A}} = \mathcal{A} - \frac{\text{div}(\mathbf{U})}{2} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{D}(\hat{\mathcal{A}}) = \mathcal{D}(\mathcal{A}),$$

IS dissipative.

The standard perturbation result in Kato ([?]) can be applied to $\hat{\mathcal{A}}$, yielding semigroup generation for the original $\mathcal{A}$.

Dissipativity

The proof of dissipativity is actually kinda cute.
It involves Green's Identities, using boundary conditions, $\text{div}(\mathbf{u}^-) = 0$, and some vector identities, to eventually get down to

$$\text{Re}\left(\hat{\mathcal{A}}\begin{bmatrix}\mathbf{u}^+\\ p^+\\ \mathbf{u}^-\end{bmatrix}, \begin{bmatrix}\mathbf{u}^+\\ p^+\\ \mathbf{u}^-\end{bmatrix}\right)_{\mathcal{H}} = -(\sigma(\mathbf{u}^+), \epsilon(\mathbf{u}^+))_{\Omega^+} - ||\nabla\mathbf{u}^-||^2_{\Omega^-} \leq 0,$$

as desired.

(This is not the hard part of the proof.)

Maximality

To show maximality of $\hat{\mathcal{A}}$ on $\mathcal{H}$, we establish the *range condition*:
$Range(\lambda I - \hat{\mathcal{A}}) = \mathcal{H}$ for λ sufficiently large.

That is, for any $[\mathbf{f}, g, \mathbf{h}] \in \mathcal{H}$, there is a solution $[\mathbf{u}^+, p^+, \mathbf{u}^-]$ to

$$(\lambda I - \hat{\mathcal{A}}) \begin{bmatrix} \mathbf{u}^+ \\ p^+ \\ \mathbf{u}^- \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \\ \mathbf{h} \end{bmatrix}.$$

Maximality (continued)

Goal: Find bilinear forms in $\mathbf{u}^-$ and p^- so we can apply the Babuska-Brezzi Theorem.

So consider $(\lambda I - \hat{\mathcal{A}}) \begin{bmatrix} \mathbf{u}^+ \\ p^+ \\ \mathbf{u}^- \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ g \\ \mathbf{h} \end{bmatrix}$, which gives the equivalent system:

$$\begin{cases} \lambda \mathbf{u}^+ + \mathbf{U} \cdot \nabla \mathbf{u}^+ - \operatorname{div} \sigma(\mathbf{u}^+) + \frac{1}{2}\operatorname{div}(\mathbf{U})\mathbf{u}^+ + \nabla p^+ = \mathbf{f} & \text{in } \Omega^+, \\ \lambda p^+ + \operatorname{div}(\mathbf{u}^+) + \mathbf{U} \cdot \nabla p^+ + \frac{1}{2}\operatorname{div}(\mathbf{U}) p^+ = g & \text{in } \Omega^+, \\ \lambda \mathbf{u}^- - \Delta \mathbf{u}^- + \nabla p^- = \mathbf{h} & \text{in } \Omega^-. \end{cases}$$

Maximality (continued)

Taking the last line, multiplying everything by $\varphi \in \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-)$, integrating over Ω^-, and applying Green's Theorems and boundary conditions gives

$$\lambda(\mathbf{u}^-, \varphi)_{\Omega^-} + (\nabla \mathbf{u}^-, \nabla\varphi)_{\Omega^-} - (p^-, \mathrm{div}(\varphi))_{\Omega^-} + \langle \sigma(\mathbf{u}^+) - p^+\vec{\nu}, \varphi\rangle_\Gamma = (\mathbf{h}, \varphi)_{\Omega^-}.$$

But the $\mathbf{u}^+$ and p^+ are still unknown ☹

Solution: Just make some more maps ☺

Maximality (continued)

Recall, need $\mathbf{u}^+, p^+$ to satisfy:

$$\begin{aligned} \lambda \mathbf{u}^+ + \mathbf{U}\cdot\nabla\mathbf{u}^+ - \operatorname{div}\sigma(\mathbf{u}^+) + \frac{1}{2}\operatorname{div}(\mathbf{U})\mathbf{u}^+ + \nabla p^+ &= \mathbf{f} \text{ in } \Omega^+, \\ \lambda p^+ + \operatorname{div}(\mathbf{u}^+) + \mathbf{U}\cdot\nabla p^+ + \frac{1}{2}\operatorname{div}(\mathbf{U})p^+ &= g \text{ in } \Omega^+, \\ \mathbf{u}^+ &= \mathbf{u}^- \text{ on } \Gamma, \\ \mathbf{u}^+ &= 0 \text{ on } \partial\Omega^+\backslash\Gamma. \end{aligned}$$

Evidently, $\mathbf{u}^+$ and p^+ depend on $\mathbf{f}$, g, and $\mathbf{u}^-$.

Maximality (continued)

So we define two maps: For $\lambda > 0$ sufficiently large,

$D_\lambda : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^1_{\partial\Omega^+ \setminus \Gamma}(\Omega^+) \times L^2(\Omega^+)$ is given by

$$D_\lambda(\varphi) = \begin{bmatrix} \mu_\lambda(\varphi) \\ q_\lambda(\varphi) \end{bmatrix},$$

where

$$\begin{cases} \lambda\mu_\lambda + \mathbf{U}\cdot\nabla\mu_\lambda - \operatorname{div}\sigma(\mu_\lambda) + \frac{1}{2}\operatorname{div}(\mathbf{U})\mu_\lambda + \nabla q_\lambda = \mathbf{0} & \text{in } \Omega^+, \\ \lambda q_\lambda + \operatorname{div}(\mu_\lambda) + \mathbf{U}\cdot\nabla q_\lambda + \frac{1}{2}\operatorname{div}(\mathbf{U}) q_\lambda = 0 & \text{in } \Omega^+, \\ \mu_\lambda|_\Gamma = \varphi & \text{on } \Gamma, \\ \mu_\lambda|_{\partial\Omega^+ \setminus \Gamma} = \mathbf{0} & \text{on } \partial\Omega^+ \setminus \Gamma. \end{cases}$$

This takes boundary values φ on Γ and maps to solutions on all of Ω^+.

Lemma

This D_λ mapping is wellposed, admitting of a unique solution with continuous dependence on data.

Maximality (continued)

Similarly, with $\mathbb{A}_\lambda : \mathbf{H}_0^1(\Omega^+) \times L^2(\Omega^+) \to \mathbf{L}^2(\Omega^+) \times L^2(\Omega^+)$ given by

$$\mathbb{A}_\lambda(\tilde{\mu}, \tilde{q}) = \begin{bmatrix} \lambda\tilde{\mu} + \mathbf{U} \cdot \nabla\tilde{\mu} - \operatorname{div} \sigma(\tilde{\mu}) + \frac{1}{2}\operatorname{div}(\mathbf{U})\tilde{\mu} + \nabla\tilde{q} \\ \lambda\tilde{q} + \operatorname{div}(\tilde{\mu}) + \mathbf{U} \cdot \nabla\tilde{q} + \frac{1}{2}\operatorname{div}(\mathbf{U})\tilde{q} \end{bmatrix},$$

we want $[\tilde{\mu}, \tilde{q}]$ such that

$$\mathbb{A}_\lambda(\tilde{\mu}, \tilde{q}) = \begin{cases} \lambda\tilde{\mu} + \mathbf{U} \cdot \nabla\tilde{\mu} - \operatorname{div} \sigma(\tilde{\mu}) + \frac{1}{2}\operatorname{div}(\mathbf{U})\tilde{\mu} + \nabla\tilde{q} = \mathbf{f} & \text{in } \Omega^+, \\ \lambda\tilde{q} + \operatorname{div}(\tilde{\mu}) + \mathbf{U} \cdot \nabla\tilde{q} + \frac{1}{2}\operatorname{div}(\mathbf{U})\tilde{q} = g & \text{in } \Omega^+, \\ \tilde{\mu} = \mathbf{0} & \text{on } \partial\Omega^+. \end{cases}$$

Thus, $[\tilde{\mu}, \tilde{q}] = \mathbb{A}_\lambda^{-1}(\mathbf{f}, g)$ takes data $[\mathbf{f}, g]$ and maps it to solutions on all of Ω^+.

Lemma

This $\mathbb{A}_\lambda$ has a bounded inverse. So the mapping $[\tilde{\mu}, \tilde{q}]$ is wellposed.

Maximality (continued)

Thus, $\begin{bmatrix} \mu_\lambda(\mathbf{u}^-) \\ q_\lambda(\mathbf{u}^-) \end{bmatrix}$ handles the condition $\mathbf{u}^+ = \mathbf{u}^-$ on Γ and $\begin{bmatrix} \tilde{\mu}(\mathbf{f}, g) \\ \tilde{q}(\mathbf{f}, g) \end{bmatrix}$ handles the non-zero right hand side $[\mathbf{f}, g]$.

So we immediately recover

$$\begin{bmatrix} \mathbf{u}^+ \\ p^+ \end{bmatrix} = \begin{bmatrix} \mu_\lambda(\mathbf{u}^-) + \tilde{\mu}(\mathbf{f}, g) \\ q_\lambda(\mathbf{u}^-) + \tilde{q}(\mathbf{f}, g) \end{bmatrix}.$$

(Note, $\mathbf{u}^-$ is still not known yet either ☹)

Maximality (continued)

Recall that we were in the middle of finding a bilinear form for $\mathbf{u}^-$ and p^-. We had

$$\lambda(\mathbf{u}^-,\varphi)_{\Omega^-}+(\nabla\mathbf{u}^-,\nabla\varphi)_{\Omega^-}-(p^-,\text{div}(\varphi))_{\Omega^-} + \langle\sigma(\mathbf{u}^+)-p^+\vec{\nu},\varphi\rangle_\Gamma = (\mathbf{h},\varphi)_{\Omega^-}.$$

With $\begin{bmatrix}\mathbf{u}^+\\ p^+\end{bmatrix} = \begin{bmatrix}\mu_\lambda(\mathbf{u}^-)+\tilde{\mu}(\mathbf{f},g)\\ q_\lambda(\mathbf{u}^-)+\tilde{q}(\mathbf{f},g)\end{bmatrix}$, this becomes

$$\lambda(\mathbf{u}^-,\varphi)_{\Omega^-}+(\nabla\mathbf{u}^-,\nabla\varphi)_{\Omega^-}-(p^-,\text{div}(\varphi))_{\Omega^-} + \langle\sigma(\mu_\lambda(\mathbf{u}^-)+\tilde{\mu}(\mathbf{f},g))-(q_\lambda(\mathbf{u}^-)+\tilde{q}(\mathbf{f},g))\vec{\nu},\varphi\rangle_\Gamma = (\mathbf{h},\varphi)_{\Omega^-}$$

for all $\varphi\in\mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-)$.

Maximality (continued)

Applying Green's Theorem to the boundary term and keeping the $\mathbf{u}^-$ terms on the left while moving the $(\mathbf{f}, g)$ terms to the right hand side, we then have

$$\lambda(\mathbf{u}^-, \varphi)_{\Omega^-} + (\nabla\mathbf{u}^-, \nabla\varphi)_{\Omega^-} - (p^-, \mathrm{div}(\varphi))_{\Omega^-} + \lambda(\mu_\lambda(\mathbf{u}^-), \mu_\lambda(\varphi))_{\Omega^-}$$
$$+ (\mathbf{U}\cdot\nabla\mu_\lambda(\mathbf{u}^-))_{\Omega^+} + \frac{1}{2}(\mathrm{div}(\mathbf{U})\mu_\lambda(\mathbf{u}^-), \mu_\lambda(\varphi))_{\Omega^+}$$
$$+ (\sigma(\mu_\lambda(\mathbf{u}^-)), \epsilon(\mu_\lambda(\varphi)))_{\Omega^+}$$
$$= (\mathbf{h}, \varphi)_{\Omega^-} + (\mathbf{f}, \mu_\lambda(\varphi))_{\Omega^+} - \Big[\lambda(\tilde{\mu}(\mathbf{f}, g), \mu_\lambda(\varphi)))_{\Omega^+}$$
$$+ (\mathbf{U}\cdot\nabla\tilde{\mu}(\mathbf{f}, g), \mu_\lambda(\varphi))_{\Omega^+} + \frac{1}{2}(\mathrm{div}(\mathbf{U})\tilde{\mu}(\mathbf{f}, g), \mu_\lambda(\varphi))_{\Omega^+}$$
$$+ (\sigma(\tilde{\mu}(\mathbf{f}, g)), \epsilon(\mu_\lambda(\varphi)))_{\Omega^+} - (\tilde{q}(\mathbf{f}, g), \mathrm{div}(\mu_\lambda(\varphi)))_{\Omega^+}\Big]$$

for all $\varphi \in \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-)$.

Additionally, from $\mathrm{div}(\mathbf{u}^-) = 0$ in Ω^-, we have

$$(\mathrm{div}(\mathbf{u}^-), \psi)_{\Omega^-} = 0 \text{ for all } \psi \in L^2(\Omega^-).$$

Maximality (continued)

Simplifying notation, we are looking for $[\mathbf{u}^-, p^-]$ that solves

$$\begin{cases} a_\lambda(\mathbf{u}^-, \varphi) + b(\varphi, p^-) = F(\varphi) & \text{for all } \varphi \in \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-) \\ b(\mathbf{u}^-, \rho) = 0 & \text{for all } \rho \in L^2(\Omega^-) \end{cases},$$

where $a_\lambda(\cdot,\cdot) : \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-) \times \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-) \to \mathbb{R}$ is given by

$$\begin{aligned} a_\lambda(\psi, \varphi) = {} & \lambda(\psi, \varphi)_{\Omega^-} + \lambda(\mu_\lambda(\psi), \mu_\lambda(\varphi))_{\Omega^+} + (\nabla\psi, \nabla\varphi)_{\Omega^-} \\ & + (\mathbf{U} \cdot \nabla\mu_\lambda(\psi), \mu_\lambda(\varphi))_{\Omega^+} + \frac{1}{2}(\mathrm{div}(\mathbf{U})\mu_\lambda(\psi), \mu_\lambda(\varphi))_{\Omega^+} \\ & + (\sigma(\mu_\lambda(\psi)), \epsilon(\mu_\lambda(\varphi)))_{\Omega^+} - (q_\lambda(\psi), \mathrm{div}(\mu_\lambda(\varphi)))_{\Omega^+}, \end{aligned}$$

$b(\cdot,\cdot) : \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-) \times L^2(\Omega^-) \to \mathbb{R}$ is given by

$$b(\varphi, \rho) = -(\rho, \mathrm{div}(\varphi))_{\Omega^-},$$

Maximality (continued)

and $F(\cdot) : \mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-) \to \mathbb{R}$ is given by

$$F(\varphi) = (\mathbf{h}, \varphi)_{\Omega^-} + (\mathbf{f}, \mu_\lambda(\varphi))_{\Omega^+} - \Big[\lambda(\tilde{\mu}(\mathbf{f}, g), \mu_\lambda(\varphi)))_{\Omega^+} + (\mathbf{U}\cdot\nabla\tilde{\mu}(\mathbf{f}, g), \mu_\lambda(\varphi))_{\Omega^+} + \frac{1}{2}(\mathrm{div}(\mathbf{U})\tilde{\mu}(\mathbf{f}, g), \mu_\lambda(\varphi))_{\Omega^+} + (\sigma(\tilde{\mu}(\mathbf{f}, g)), \epsilon(\mu_\lambda(\varphi)))_{\Omega^+} - (\tilde{q}(\mathbf{f}, g), \mathrm{div}(\mu_\lambda(\varphi)))_{\Omega^+}\Big].$$

Maximality (continued)

For the Inf-Sup condition, we invoke a lemma from [?]:

Lemma (Grisvard)

For $\Omega \subset \mathbb{R}^n$ that is bounded, open, and with Lipshitz boundary $\partial\Omega$, there exists some $\delta > 0$ and $\mu \in [C^\infty(\bar{\Omega})]^n$ such that $\mu \cdot \vec{\nu} \geq \delta$ a.e. on $\partial\Omega$.

With this in hand, let $\omega \in \mathbf{H}^1_{\partial\Omega^- \setminus \Gamma}(\Omega^-)$ be a solution to

$$\begin{cases} \text{div}(\omega) = -\eta \langle \mu, \vec{\nu} \rangle_\Gamma & \text{in } \Omega^-, \\ \omega|_{\partial\Omega^- \setminus \Gamma} = 0 & \text{on } \partial\Omega^- \setminus \Gamma, \\ \omega|_\Gamma = \left(\int_{\Omega^-} \eta \, d\Omega^- \right) \mu(x) & \text{on } \Gamma, \end{cases}$$

for any $\eta \in L^2(\Omega^-)$. It is well-known that solution, ω, exists with $||\nabla\omega||_{\Omega^-} \leq C||\eta||_{\Omega^-}$.

Maximality (continued)

Now consider

$$\sup_{\varphi\in\mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-)} \frac{b(\varphi,\eta)}{||\varphi||_{\mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-)}} \overset{\text{Poincaré's}}{=} \sup_{\varphi\in\mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}} \frac{b(\varphi,\eta)}{||\nabla\varphi||_{\Omega^-}}$$

$$(b(\varphi,\eta) = -(\eta,\mathrm{div}(\varphi))_{\Omega^-}) = \sup_{\varphi\in\mathbf{H}^1_{\partial\Omega^-\setminus\Gamma}(\Omega^-)} \frac{-\int \eta\,\mathrm{div}(\varphi)\,d\Omega^-}{||\nabla\varphi||_{\Omega^-}}$$

$$\geq \frac{-\int \eta\,\mathrm{div}(\omega)\,d\Omega^-}{||\nabla\omega||_{\Omega^-}}$$

$$(\mathrm{div}(\omega) = -\eta\langle\mu,\vec{\nu}\rangle_\Gamma) = \frac{\int \eta^2\langle\mu,\vec{\nu}\rangle_\Gamma\,d\Omega^-}{||\nabla\omega||_{\Omega^-}}$$

$$(\mu\cdot\vec{\nu}\geq\delta) \geq \frac{\delta\cdot\mathrm{meas}(\Gamma)||\eta||^2_{\Omega^-}}{||\nabla\omega||_{\Omega^-}}$$

$$(||\nabla\omega||_{\Omega^-}\leq C||\eta||_{\Omega^-}) \geq \frac{\delta\cdot\mathrm{meas}(\Gamma)||\eta||_{\Omega^-}\left(\frac{1}{C}||\nabla\omega||_{\Omega^-}\right)}{||\nabla\omega||_{\Omega^-}}$$

$$= \left(\frac{1}{C}\,\delta\,\mathrm{meas}(\Gamma)\right)||\eta||_{\Omega^-}.$$

Maximality (continued)

Thus, we have

$$\sup_{\varphi \in \mathbf{H}^1_{\partial\Omega^- \backslash \Gamma}(\Omega^-)} \frac{b(\varphi, \eta)}{||\varphi||_{\mathbf{H}^1_{\partial\Omega^- \backslash \Gamma}(\Omega^-)}} \geq \beta ||\eta||_{\Omega^-},$$

and since $\eta \in L^2(\Omega^-)$ was arbitrary,

$$\inf_{\eta \in L^2(\Omega^-)} \sup_{\varphi \in \mathbf{H}^1_{\partial\Omega^- \backslash \Gamma}(\Omega^-)} \frac{b(\varphi, \eta)}{||\eta||_{\Omega^-} ||\varphi||_{\partial\Omega^- \backslash \Gamma}} \geq \beta,$$

with $\beta = \frac{1}{C}\, \delta\, \text{meas}(\Gamma)$. So the Inf-Sup condition is satisfied.

Conclusion of Proof

Thus, by the Babuska-Brezzi Theorem, we have the desired solutions $[\mathbf{u}^-, p^-]$. Along the way, we found maps which gave us $\begin{bmatrix} \mathbf{u}^+ \\ p^+ \end{bmatrix} = \begin{bmatrix} \mu_\lambda(\mathbf{u}^-) + \tilde{\mu}(\mathbf{f}, g) \\ q_\lambda(\mathbf{u}^-) + \tilde{q}(\mathbf{f}, g) \end{bmatrix}$. (These establish Part (ii) of Theorem.)

After showing $[\mathbf{u}^+, p^+, \mathbf{u}^-] \in \mathcal{D}(\mathcal{A})$, we have established maximality of $\hat{\mathcal{A}}$, which allows us to use Lumer-Phillips Theorem to give us a C_0-semigroup of contractions. (This established Part (i) of Theorem.)

□

The Finite Element Method

Domain is discretized in to a *mesh* with *elements* and *nodes*.

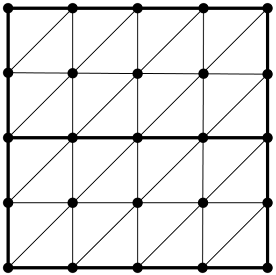
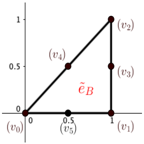
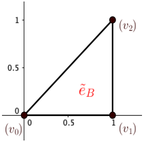


Figure: A sample mesh.



Fluid velocity reference element



Pressure reference element

The Discrete Problem

FEM idea:
Assume $\mathbf{u} = \sum_{i=1}^{N} \vec{\alpha}_i \vec{\varphi}_i(x,y)$ for known basis functions $\{\varphi_i\}_{i=1}^{N}$ and $p = \sum_{i=1}^{N_p} \beta_i \psi_i$ for basis functions $\{\psi_i\}_{i=1}^{N_p}$.
Then just need to find α_i's and β_i's.

The variational form from before

$$a_\lambda(\mathbf{u}^-, \varphi) + b(\varphi, p) = F(\varphi) \text{ for all } \varphi \in \mathbf{H}^1_{\partial\Omega^- \setminus \Gamma}(\Omega^-)$$
$$b(\mathbf{u}^-, q) = 0 \qquad \text{for all } q \in L^2(\Omega^-)$$

lends itself to the matrix equation

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \beta \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix}.$$

We use similar formulation to find $[\mu_\lambda, q_\lambda]$ and $[\tilde{\mu}, \tilde{q}]$.

Numerical Test Problem

Take $\mathbf{U} = 0$, $\Omega^+ = (1,0) \times (.5,1)$, and $\Omega^- = (0,1) \times (0,.5)$.

Then

$$\mathbf{u}^+ = \begin{bmatrix} 2\sin(2\pi x)\cos(2\pi y) \\ \cos(2\pi x)\sin(2\pi y) \end{bmatrix}, \quad \mathbf{u}^- = \begin{bmatrix} 2\sin(2\pi x)\cos(2\pi y) \\ -2\cos(2\pi x)\sin(2\pi y) \end{bmatrix}$$

$$p^+ = 2\pi(2\nu + 3\lambda - 2)\cos(2\pi x), \quad p^- = 0$$

solve our system for right hand side data

$$\begin{aligned} \mathbf{f} &= \lambda \mathbf{u}^+ - \text{div}\sigma(\mathbf{u}^+) + \nabla p^+ \\ &= \begin{bmatrix} (2\lambda + 16\nu\pi^2 + 12(\nu + \tilde{\lambda}))\sin(2\pi x)\cos(2\pi y) \\ (\lambda + 8\nu\pi^2 + 12(\nu + \tilde{\lambda}))\cos(2\pi x)\sin(2\pi y) \end{bmatrix}, \\ g &= \lambda p^+ + \text{div}(\mathbf{u}^+) \\ &= 2\pi\lambda(2\nu + 3\tilde{\lambda} - 2)\cos(2\pi x) + 6\pi\cos(2\pi x)\cos(2\pi y), \\ \mathbf{h} &= \lambda \mathbf{u}^- - \triangle \mathbf{u}^- + \nabla p^- \\ &= \begin{bmatrix} (2\lambda + 16\pi^2)\sin(2\pi x)\cos(2\pi y) \\ -(2\lambda + 16\pi^2)\cos(2\pi x)\sin(2\pi y) \end{bmatrix}. \end{aligned}$$

Numerical Results

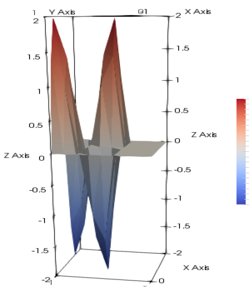
For this problem, the errors in FEM approximations are given below.

# elements in Ω^+	Side length	$\|\mathbf{u}^+ - \mathbf{u}_h^+\|_0$	$\|\mathbf{u}^+ - \mathbf{u}_h^+\|_1$	$\|p^+ - p_h^+\|_0$
4	0.5	5.158	0.280	.783
16	0.25	1.533	0.0497	1.107
64	0.125	0.413	5.89×10^{-3}	0.232
256	0.0625	0.106	7.25×10^{-4}	0.055
1024	0.03125	0.0266	9.04×10^{-5}	0.0136

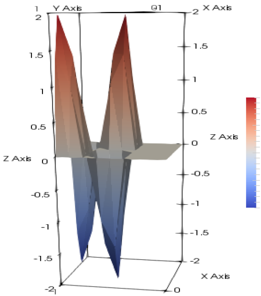
# elements in Ω^-	Side length	$\|\mathbf{u}^- - \mathbf{u}_h^-\|_0$	$\|\mathbf{u}^- - \mathbf{u}_h^-\|_1$	$\|p^- - p_h^-\|_0$
4	0.5	6.715	0.296	3.053
16	0.25	1.907	0.059	0.404
64	0.125	0.519	7.17×10^{-3}	0.032
256	0.0625	0.134	9.06×10^{-4}	2.44×10^{-3}
1024	0.03125	0.033	1.14×10^{-4}	1.92×10^{-4}

Numerical Results (continued)

Since $\mathbf{u}^+$ and $\mathbf{u}^-$ are vector valued, we compare plots of approximate and true solutions for each component. Images shown are with 64 elements in domain.



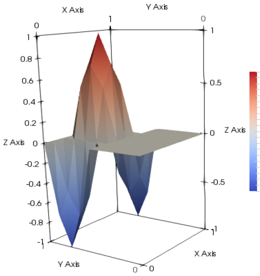
Approximate u_1^+



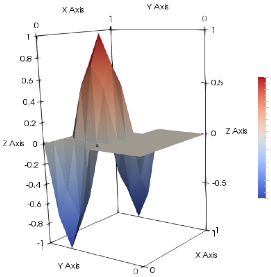
True u_1^+

Numerical Results (continued)

For u_2^+:



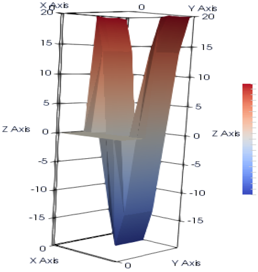
Approximate u_2^+



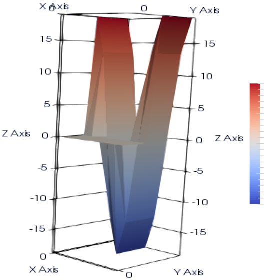
True u_2^+

Numerical Results (continued)

For p^+:



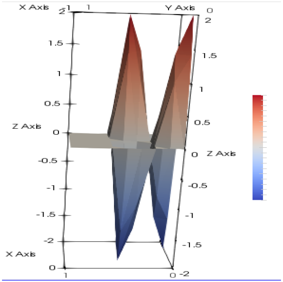
Approximate p^+



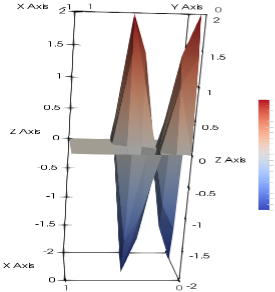
True p^+

Numerical Results (continued)

For u_1^-:



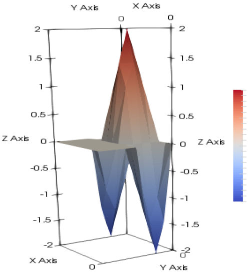
Approximate u_1^-



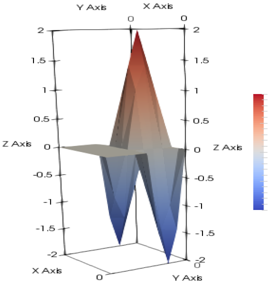
True u_1^-

Numerical Results (continued)

For u_2^-:



Approximate u_2^-



True u_2^-