

Monday, 2/5/2024

Enumerative Geometry of Gauged Linear Sigma Models I GLSM

Main References

[FJR] = Fan-Jarvis-Ruan, "A mathematical theory of the gauged linear sigma models"

[FK] = D. Favero & B. Kim, "General GLSM invariants and their Cohomological Field Theories
CohFTs"

§1 Geometry of GLSMs

The input data of a GLSM is a 5-tuple

$$(V, G, \mathbb{C}_R^*, W, \theta)$$

(1) (linear space) $V = \text{Spec } \mathbb{C}[X_1, \dots, X_N] = \mathbb{C}^N$

(2) (gauge group) $G \subset GL(V)$

(3) (R-symmetries)

$$\mathbb{C}_R^* \cong \mathbb{C}^*$$

acts on V by weights $c_1, \dots, c_N \in \mathbb{Z}$

Assume G, \mathbb{C}^* commute, $G \cap \mathbb{C}^* = \langle J \rangle \cong \mu_r$ r -th roots of unity
 $\Gamma := G \mathbb{C}^* \subset GL(V)$ $\text{diag}(e^{2\pi i c_1/r}, \dots, e^{2\pi i c_N/r})$

\Rightarrow short exact sequence

$$1 \rightarrow G \rightarrow \Gamma \xrightarrow{\chi} \mathbb{C}^* \rightarrow 1$$

\parallel
 $\Gamma/G = \mathbb{C}^*/\langle J \rangle$

(4) (super-potential) $W: V \rightarrow \mathbb{C}$ polynomial function
 $W(\tau \cdot x) = \chi(\tau) W(x)$

In particular, $W \in \mathbb{C}[x_1, \dots, x_N]^G$

$$t \in \mathbb{C}^* \quad W(t^{c_1} x_1, \dots, t^{c_N} x_N) = t^r W(x_1, \dots, x_N)$$

(5) (stability condition) $\theta \in \hat{G} := \text{Hom}(G, \mathbb{C}^*)$

G -polarization $L_\theta \rightarrow V$

$$V_G^{ss}(\theta) = \{v \in V : \exists k \in \mathbb{Z}_{>0} \text{ and } f \in H^0(V, L_\theta^k)^G, \pi(v) \neq 0\}$$

where $H^0(V, L_\theta^k)^G = \left\{ \begin{array}{l} f \in \mathbb{C}[x] \\ \parallel \\ \mathbb{C}[x_1, \dots, x_N] \end{array} \right\}$, $f(\tau \cdot v) = \theta(\tau)^k f(v)$
 $\forall \tau \in G \forall v \in V$

Assume $V_G^{ss}(\theta) = V_G^s(\theta)$

$$\Rightarrow X_\theta = [V //_\theta G] := [V_G^{SS}(\theta) / G] \quad \text{GIT stack quotient}$$

Smooth Deligne-Mumford (DM)
stack with trivial generic
stabilizer i.e. orbifold

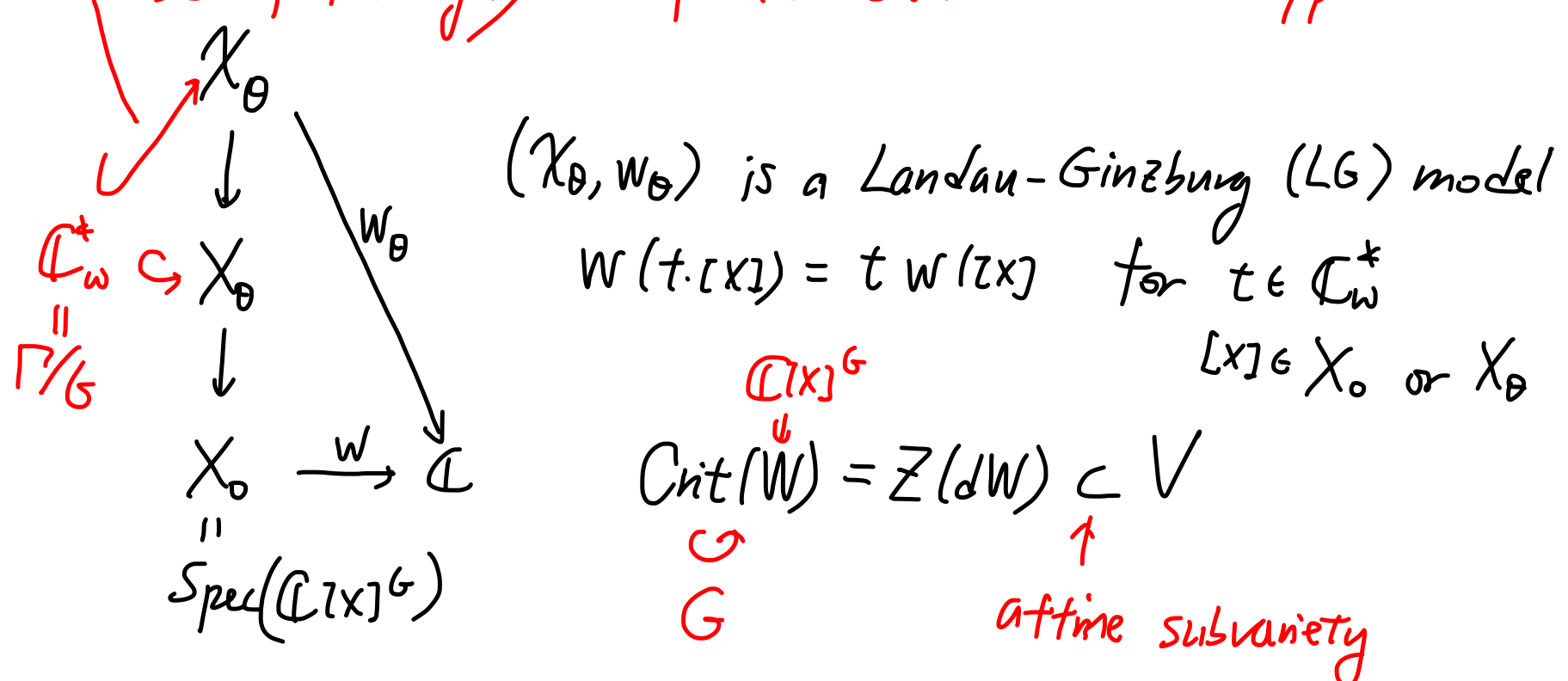
$$\downarrow$$

$$X_\theta = V //_\theta G := \text{Proj} \left(\bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(V, L_\theta^k) \right)^G \quad \text{GIT quotient}$$

↓ projective

$$X_0 = V /_{\text{aff}} G := \text{Spec}(\mathbb{C}[x]^G) \quad \text{affine quotient}$$

in the sense of Romagny "Group actions on stacks and applications"



Assume $\mathcal{Z}_\theta := [Z(dW) //_\theta G]$ proper, can be singular
 $\Rightarrow Z_\theta := Z(dW) //_\theta G$ projective over $\bullet = \text{Spec } \mathbb{C}$

GLSM invariants of $\underline{X} = (V, G, \mathbb{C}^*, W, \theta)$
 are virtual counts of curves in $\mathcal{Z}_\theta = \mathbb{Z}(dW_\theta)$

When $\mathcal{Z}_\theta = \mathbb{Z}_\theta$ is a smooth projective variety,
 GLSM invariants of \underline{X} are Gromov-Witten (GW)
 invariants of \mathcal{Z}_θ (up to sign)

More generally, when \mathcal{Z}_θ is a smooth DM stack,
 GLSM invariants of \underline{X} are orbifold GW invariants of \mathcal{Z}_θ .

Example $V = \text{Spec } \mathbb{C}[x_1, \dots, x_5, p] = \mathbb{C}^6$

$G = \mathbb{C}^*$ acts on V by weights $(1, 1, 1, 1, 1, -5)$ gauge charges
 \mathbb{C}_R^* " $(0, 0, 0, 0, 0, 1)$

$G \cap \mathbb{C}_R^*$ is trivial, $\mathbb{C}_R^* = \mathbb{C}_W^*$

$W = pW_5(x)$, where $W_5(x) = \sum_{i=1}^5 x_i^5$ Fermat quintic
 polynomial
 (x_1, \dots, x_5)

$$\theta \in \hat{G} \cong \mathbb{Z}$$

$$s^k \hookrightarrow k$$

$$s \in G$$

Calabi-Yan

$\theta > 0$ (CY phase)

$$V_G^{ss}(\theta) = (\mathbb{C}^5 - \{0\}) \times \mathbb{C}$$

$$\begin{aligned} X_\theta = X_\theta &= ((\mathbb{C}^5 - \{0\}) \times \mathbb{C}) / G \\ &= \text{Tot}(\mathcal{O}_{\mathbb{P}^4}(-5)) = K_{\mathbb{P}^4} \end{aligned}$$

$$\begin{aligned} Z_\theta = Z_\theta &= X_5 \subset \mathbb{P}^4 \subset K_{\mathbb{P}^4} \\ &\quad \downarrow \quad \downarrow \\ &\quad \{W_5(x) = 0 = p\} \quad \{p=0\} \end{aligned}$$

GLSM invariants

= \pm GW invariants of X_5

quintic CY 3-fold

$\theta < 0$ (LG phase)

$$V_G^{ss}(\theta) = \mathbb{C}^5 \times (\mathbb{C} - \{0\})$$

$$\begin{aligned} X_\theta &= [(\mathbb{C}^5 \times (\mathbb{C} - \{0\})) / G] \\ &= [(\mathbb{C}^5 \times \{1\}) / \mu_5] \cong [\mathbb{C}^5 / \mu_5] \end{aligned}$$

$$X_\theta = \mathbb{C}^5 / \mu_5$$

$$Z_\theta = \{x_i^4 = 0\} \subset [\mathbb{C}^5 / \mu_5]$$

GLSM invariants

= FJRW invariants of (W_5, μ_5)

Fan-Jarvis-Ruan-Witten

LG/CY Correspondence

§ 2 Moduli of LG quasimaps

A genus g , l -pointed LG quasimap to $\underline{X} = (V, G, \mathbb{C}^X, W, \theta)$ is a 4-tuple $\underline{u} = (\mathbb{C}, z_1, \dots, z_l), P, \mathbb{K}, u)$ where

(1) $(\mathbb{C}, z_1, \dots, z_l)$ genus g , l -pointed twisted curve nodal orbicurve, z_1, \dots, z_l smooth distinct points on \mathbb{C}

$(\mathbb{C}, z_1, \dots, z_l) \xrightarrow{\gamma} (\mathbb{C}, z_1, \dots, z_l)$ coarse curve isomorphism outside nodes and marking

(C, z_1, \dots, z_l) genus g , l -pointed prestable curve

$$z_i \quad 0 \in [\mathbb{C}/\mu_r] \quad \sum_{\substack{p \\ \mu_r}} z = \sum z$$

node $0 \in \left[\text{Spec}(\mathbb{C}[x, y] / \langle xy \rangle) / \mu_r \right] \quad \sum \cdot (x, y) = (\sum x, \sum y)$

(2) $P \rightarrow C$ principal Γ -bundle

$$\begin{array}{ccc} \text{scheme} \leftarrow P & \longrightarrow & \bullet \\ \downarrow \square & & \downarrow \\ [P/\Gamma] = C & \xrightarrow{\quad} & B\Gamma \end{array} \quad \begin{array}{l} \text{by } y \in C \quad \text{Aut}(y) \rightarrow \Gamma \text{ injective} \\ \uparrow \\ \text{representable} \end{array}$$

Notation $\phi: \Gamma \rightarrow GL(E)$ ^{→ vector space / \mathbb{C}} representation of Γ

$P \times_{\phi} E := [(P \times E) / \Gamma]$ where Γ acts on $P \times E$ on the right by

$$(p, v) \cdot \gamma = (p \cdot \gamma, \phi(\gamma^{-1})v) \quad \gamma \in \Gamma, p \in P, v \in E$$

(3) $\kappa: P \times_{\omega} \mathbb{C} \xrightarrow{\cong} \omega_C^{\text{Log}} = \omega_C(z_1 + \dots + z_l) = \text{pullback of}$

$$\Gamma \times_{\omega} \mathbb{C}^* \subseteq GL(\mathbb{C})$$

$$\omega_C^{\text{Log}}(z_1 + \dots + z_l) \text{ under } C \rightarrow C \downarrow \text{coarse curve}$$

Leg $2g-2+l$

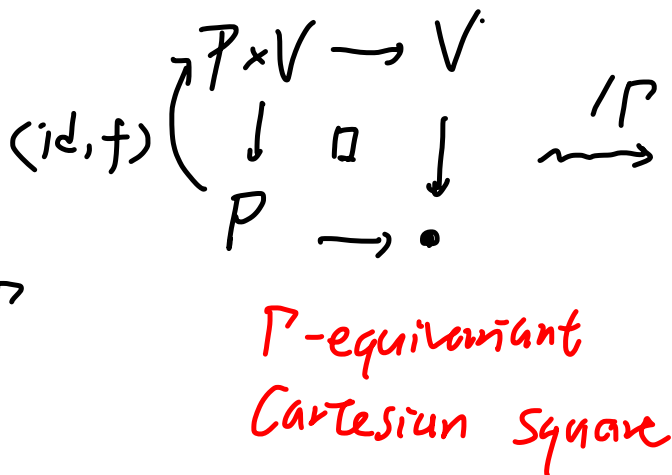
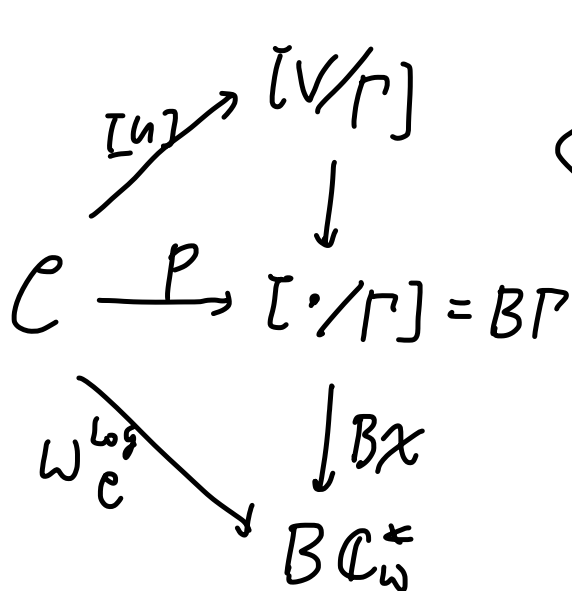
(P, \mathcal{K}) is known as a Γ -structure on $(\mathbb{C}, z_1, \dots, z_\ell)$

(4) $u \in H^0(P \times V)$

$\rho: \Gamma \rightarrow GL(V)$

$B(u) := u^{-1}(P \times V_G^{un}/\theta)$, $V_G^{un}/\theta = V \setminus V_G^{ss}/\theta$

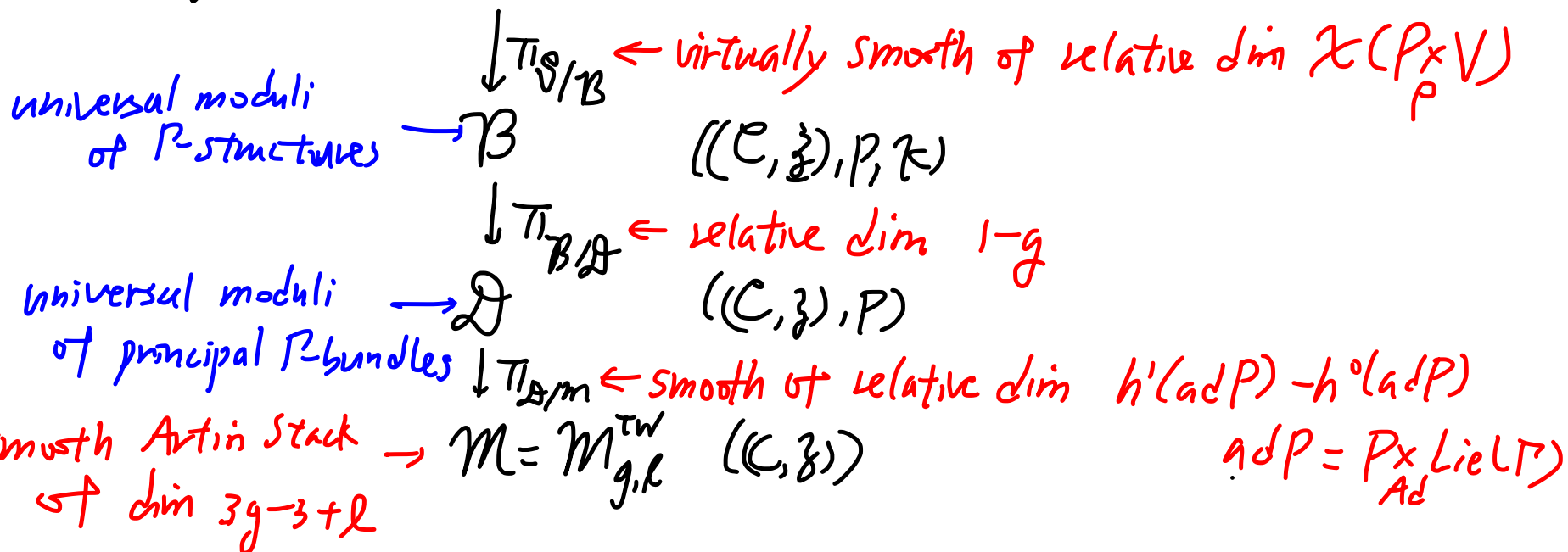
\uparrow
 finite, away from nodes and markings



$1 \rightarrow G \rightarrow \Gamma \xrightarrow{\mathcal{X}} \mathbb{C}_{\omega}^* \rightarrow 1$

$LG_{g,\ell}(\underline{z})$ moduli of genus g , ℓ -marked LG quasimaps to \underline{z}

$LG_{g,\ell}(\underline{z}) \subset^{open} \mathcal{S} \quad ((\mathbb{C}, \underline{z}), P, \mathcal{K}, u)$ without \star



$\mathcal{M}, \mathcal{D}, \mathcal{B}$ smooth Artin stacks

Exercise Compute the virtual dimension of $LG_{g,l}(\underline{\lambda})$

$$d_{\text{vir}}^{LG_{g,l}(\underline{\lambda})} = 2g-2+l + \chi(P_{\rho} \times V) - \chi(\text{ad}P)$$

locally constant on $LG_{g,l}(\underline{\lambda})$

[FJR] Lemma 6.1.7.

$$\begin{array}{ccccccc}
 P_S & \rightarrow & P_B & \rightarrow & P_D & \leftarrow & \text{universal principal bundle} \\
 \downarrow & \square & \downarrow & \square & \downarrow & & \\
 C_S & \rightarrow & C_B & \rightarrow & C_D & \rightarrow & C_M \leftarrow \text{universal curve} \\
 \downarrow \pi_S & \square & \downarrow \pi_B & \square & \downarrow \pi_D & \square & \downarrow \pi_M \\
 S & \xrightarrow{\pi_{S/B}} & B & \xrightarrow{\pi_{B/D}} & D & \xrightarrow{\pi_{D/M}} & M
 \end{array}$$

$$\begin{array}{ccccccc}
 \mathcal{U}_S & \rightarrow & \mathcal{U}_B & \rightarrow & \mathcal{U}_D = P_D \times_{\rho} V \\
 \downarrow & \square & \downarrow & \square & \downarrow \\
 C_S & \rightarrow & C_B & \rightarrow & C_D \\
 \downarrow \pi_S & \square & \downarrow \pi_B & \square & \downarrow \pi_D \\
 S & \xrightarrow{\pi_{S/B}} & B & \xrightarrow{\pi_{B/D}} & D
 \end{array}$$

universal section \mathcal{U}_S

relative perfect obstruction theory

$$E_{S/B} = (R\pi_{S*} \mathcal{U}_S)^\vee$$