Monday, 2/5/2024

Enumerative Geometry of Gauged Linear Sigma Models I

GLSM

Main References

[FJR] = Fan-Jarvis-Ruan, "A mathematical theory of the gauged linear sigma models"

[FK] = D. Favero & B. Kim, "General GLSM invariants and their Cohomological Field Theories"

CohFTs

§1 Geometry of GLSMs

The input data of a GLSM is a 5-tuple

$(V, G, \mathbb{C}^*_R, W, \theta)$

(1) (linear space) $\quad V = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_N] = \mathbb{C}^N$

(2) (gauge group) $\quad G \subset GL(V)$

$$\cup$$

(3) (R-symmetries) $\quad \mathbb{C}^*_R \cong \mathbb{C}^*$

acts on V by weights $c_1, \ldots, c_N \in \mathbb{Z}$

r-th roots of unity

Assume $G, \mathbb{C}_R^*$ commute, $G \cap \mathbb{C}_R^* = \langle J \rangle \cong \mu_r$

$J = \mathrm{diag}(e^{2\pi i c_1/r}, \dots, e^{2\pi i c_N/r})$

$\Gamma := G\mathbb{C}_R^* \subset GL(V)$

$\Rightarrow$ short exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{C}_\omega^* \to 1$$

$$\mathbb{C}_\omega^* = \Gamma/G = \mathbb{C}_R^*/\langle J \rangle$$

(4) (superpotential) $W : V \to \mathbb{C}$ polynomial function

$$W(\gamma \cdot x) = \chi(\gamma) W(x)$$

In particular, $W \in \mathbb{C}[x_1, \dots, x_N]^G$

$t \in \mathbb{C}_R^*$ $\quad W(t^{c_1}x_1, \dots, t^{c_N}x_N) = t^r W(x_1, \dots, x_N)$

(5) (stability condition) $\theta \in \hat{G} := \mathrm{Hom}(G, \mathbb{C}^*)$

G-polarization $L_\theta \to V$

$$V_G^{ss}(\theta) = \{ v \in V : \exists k \in \mathbb{Z}_{>0} \text{ and } f \in H^0(V, L_\theta^k)^G, f(v) \neq 0 \}$$

where $H^0(V, L_\theta^k)^G = \{ f \in \mathbb{C}[x],\ f(\gamma \cdot v) = \theta(\gamma)^k f(v)\ \forall \gamma \in G\ \forall v \in V \}$

$\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_N]$

Assume $V_G^{ss}(\theta) = V_G^s(\theta)$

$$\Rightarrow \quad \mathcal{X}_\theta = [V/\!/_\theta G] := [V_G^{ss}(\theta)/G]$$ GIT stack quotient

Smooth Deligne-Mumford (DM) stack with trivial generic stabilizer i.e. orbifold

$$\downarrow$$

$$X_\theta = V/\!/_\theta G := \mathrm{Proj}\Big(\bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(V, L_\theta^k)^G\Big)$$ GIT quotient

$\downarrow$ projective

$$X_0 = V/_{\mathrm{aff}}\, G := \mathrm{Spec}(\mathbb{C}[x]^G)$$ affine quotient

in the sense of Romagny "Group actions on stacks and applications"

$$\begin{tikzcd}
 & \mathcal{X}_\theta \arrow[d] \arrow[ddr, "w_\theta"] & \\
\mathbb{C}^*_\omega \arrow[ur] \arrow[r, phantom, "\circlearrowright"] & X_\theta \arrow[d] & \\
 & X_0 \arrow[r, "W"] & \mathbb{C}
\end{tikzcd}$$

$\mathbb{C}^*_\omega = \Gamma/G$

$X_0 = \mathrm{Spec}(\mathbb{C}[x]^G)$

$(\mathcal{X}_\theta, w_\theta)$ is a Landau-Ginzburg (LG) model

$W(t\cdot[x]) = t\,W([x])$ for $t\in\mathbb{C}^*_\omega$

$[x]\in X_0$ or X_θ

$\mathbb{C}[x]^G$

$$\mathrm{Crit}(W) = Z(dW) \subset V$$

$G \circlearrowright \mathrm{Crit}(W)$

affine subvariety

<u>Assume</u> $\mathfrak{Z}_\theta := [Z(dW)/\!/_\theta G]$ proper, can be singular

$\Rightarrow$ $Z_\theta := Z(dW)/\!/_\theta G$ projective over $\bullet = \mathrm{Spec}\,\mathbb{C}$

GLSM invariants of $\underline{\mathfrak{X}} = (V, G, \mathbb{C}_R^*, W, \theta)$ are virtual counts of curves in $\mathfrak{Z}_\theta = Z(dW_\theta)$

When $\mathfrak{Z}_\theta = Z_\theta$ is a smooth projective variety, GLSM invariants of $\underline{\mathfrak{X}}$ are Gromov-Witten (GW) invariants of $\mathfrak{Z}_\theta$ (up to sign)

More generally, when $\mathfrak{Z}_\theta$ is a smooth DM stack, GLSM invariants of $\underline{\mathfrak{X}}$ are orbifold GW invariants of $\mathfrak{Z}_\theta$.

<u>Example</u> $V = \mathrm{Spec}\,\mathbb{C}[x_1, \dots x_5, p] = \mathbb{C}^6$

$G = \mathbb{C}^*$ acts on V by weights $(1,1,1,1,1,-5)$ gauge charges

$\mathbb{C}_R^*$ 〃 $(0,0,0,0,0,1)$

$G \cap \mathbb{C}_R^*$ is trivial, $\mathbb{C}_R^* = \mathbb{C}_\omega^*$

$W = p\,W_5(x)$, where $W_5(x) = \sum_{i=1}^{5} x_i^5$ Fermat quintic polynomial

$x = (x_1, \dots, x_5)$

$\theta \in \hat{G} \cong \mathbb{Z}$

$s^k \longleftarrow\!\shortmid\, k$

$s \in G$

Calabi-Yau

<u>$\theta > 0$ (CY phase)</u>

$V_G^{ss}(\theta) = (\mathbb{C}^5 - \{0\}) \times \mathbb{C}$

$\mathcal{X}_\theta = X_\theta = ((\mathbb{C}^5 - \{0\}) \times \mathbb{C}) / G$

$= \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^4}(-5)) = K_{\mathbb{P}^4}$

$\mathfrak{Z}_\theta = Z_\theta = X_5 \subset \mathbb{P}^4 \subset K_{\mathbb{P}^4}$

$\{W_5(x) = 0 = p\}$ $\qquad$ $\{p = 0\}$

GLSM invariants

$= \pm$ GW invariants of X_5

quintic CY 3-fold

<u>$\theta < 0$ (LG phase)</u>

$V_G^{ss}(\theta) = \mathbb{C}^5 \times (\mathbb{C} - \{0\})$

$\mathcal{X}_\theta = [(\mathbb{C}^5 \times (\mathbb{C} - \{0\})) / G]$

$= [(\mathbb{C}^5 \times \{1\}) / \mu_5] \cong [\mathbb{C}^5 / \mu_5]$

$X_\theta = \mathbb{C}^5 / \mu_5$

$\mathfrak{Z}_\theta = \{x_i^4 = 0\} \subset [\mathbb{C}^5 / \mu_5]$

GLSM invariants

$=$ FJRW invariants of (W_5, μ_5)

Fan-Jarvis-Ruan-Witten

LG/CY correspondence

<u>§2 Moduli of LG quasimaps</u>

A genus g, ℓ-pointed LG quasimap to $\underline{\mathcal{X}} = (V, G, \mathbb{C}^*_R, W, \theta)$ is a 4-tuple $\underline{u} = ((\mathcal{C}, \mathfrak{z}_1, \ldots, \mathfrak{z}_\ell), P, \kappa, u)$ where

(1) $(\mathcal{C}, \mathfrak{z}_1, \ldots, \mathfrak{z}_\ell)$ genus g, ℓ-pointed twisted curve

nodal orbicurve, $\mathfrak{z}_1, \ldots, \mathfrak{z}_\ell$ smooth distinct points on $\mathcal{C}$

$(\mathcal{C}, \mathfrak{z}_1, \ldots, \mathfrak{z}_\ell) \to (C, z_1, \ldots, z_\ell)$ coarse curve

isomorphism outside nodes and marking

$(C, z_1, \dots, z_\ell)$ genus-g, ℓ-pointed prestable curve

$\mathfrak{z}_i \qquad 0 \in [\mathbb{C}/\mu_r] \qquad \underset{\underset{\mu_r}{\uparrow}}{\zeta} \cdot z = \zeta z$

node $\qquad 0 \in \left[\mathrm{Spec}\left(\mathbb{C}[x,y]/\langle xy\rangle\right)/\mu_r\right] \qquad \zeta\cdot(x,y) = (\zeta x, \zeta^{-1} y)$

(2) $\quad \mathcal{P} \to \mathcal{C}$ principal Γ-bundle

$$\begin{tikzcd} \text{scheme} & P \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bullet \arrow[d] \\ & {[P/\Gamma] = \mathcal{C}} \arrow[r, "\text{representable}"'] & B\Gamma \end{tikzcd}$$

$\forall y \in \mathcal{C} \quad \mathrm{Aut}(y) \to \Gamma$ injective

<u>Notation</u> $\quad \phi: \Gamma \to GL(\underset{\text{vectorspace}/\mathbb{C}}{E})$ representation of Γ

$P \times_\phi E := [(P \times E)/\Gamma]$ where Γ acts on $P \times E$ on the right by

$(p, v)\cdot\gamma = (p\cdot\gamma, \phi(\gamma^{-1})v) \qquad \gamma \in \Gamma, \ p \in P, \ v \in E$

(3) $\kappa: P \times_\chi \mathbb{C} \xrightarrow{\cong} \omega_{\mathcal{C}}^{\log} = \omega_{\mathcal{C}}(\mathfrak{z}_1 + \dots + \mathfrak{z}_\ell)$ = pullback of

$\Gamma \xrightarrow{\chi} \mathbb{C}^*_\omega \subseteq GL(\mathbb{C}) \qquad \omega_C^{\log}(z_1 + \dots + z_\ell)$ under $\mathcal{C} \to C$

deg $2g-2+\ell$

C: coarse curve

(P, κ) is known as a Γ-structure on $(C, z_1, \dots, z_\ell)$

(4) $u \in H^0(P \times_\rho V)$ $\qquad B(u) := u^{-1}(P \times_\rho V_G^{un}(\theta))$, $V_G^{un}(\theta) = V \setminus V_G^{ss}(\theta)$

$\rho : \Gamma \to GL(V)$

✲ finite, away from nodes and markings

$$\begin{tikzcd} & {[V/\Gamma]} \arrow[d] \\ C \arrow[ur, "{[u]}"] \arrow[r, "P"] \arrow[dr, "\omega_C^{\log}"'] & {[\cdot/\Gamma] = B\Gamma} \arrow[d, "B\chi"] \\ & B\mathbb{C}^*_\omega \end{tikzcd}$$

$$\begin{tikzcd} P \times V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ P \arrow[r] \arrow[u, bend left, "{(id, f)}"] & \bullet \end{tikzcd} \quad \xrightarrow{/\Gamma} \quad \begin{tikzcd} P \times_\rho V \arrow[r] \arrow[d] & {[V/\Gamma]} \arrow[d] \\ C \arrow[r] \arrow[u, bend left, "u"] \arrow[ur, "{[u]}"] & B\Gamma \end{tikzcd}$$

Γ-equivariant Cartesian square

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{C}^*_\omega \to 1$$

$LG_{g,\ell}(\mathcal{X})$ moduli of genus g, ℓ-marked LG quasimaps to $\mathcal{X}$

$LG_{g,\ell}(\mathcal{X}) \overset{\text{open}}{\subset} \mathcal{S}$ $\qquad ((C, \underline{z}), P, \kappa, u)$ without ✲ $\qquad \underline{z} = (z_1, \dots, z_\ell)$

$\downarrow \pi_{\mathcal{S}/\mathcal{B}}$ ← virtually smooth of relative dim $\chi(P \times_\rho V)$

universal moduli of Γ-structures → $\mathcal{B}$ $\qquad ((C, \underline{z}), P, \kappa)$

$\downarrow \pi_{\mathcal{B}\mathcal{D}}$ ← relative dim $1-g$

universal moduli of principal Γ-bundles → $\mathcal{D}$ $\qquad ((C, z), P)$

$\downarrow \pi_{\mathcal{D}/\mathcal{M}}$ ← smooth of relative dim $h^1(\mathrm{ad}P) - h^0(\mathrm{ad}P)$

smooth Artin stack of dim $3g-3+\ell$ → $\mathcal{M} = \mathcal{M}_{g,\ell}^{tw}$ $\qquad ((C, z))$ $\qquad \mathrm{ad}P = P \times_{Ad} \mathrm{Lie}(\Gamma)$

$\mathcal{M}, \mathcal{D}, \mathcal{B}$ smooth Artin stacks

<u>Exercise</u> Compute the virtual dimension of $LG_{g,\ell}(\underline{\mathfrak{X}})$

$$d^{\mathrm{vir}}_{LG_{g,\ell}(\underline{\mathfrak{X}})} = 2g-2+\ell + \chi(P\times_\rho V) - \chi(\mathrm{ad}P)$$

locally constant on $LG_{g,\ell}(\underline{\mathfrak{X}})$

[FJR] Lemma 6.1.7.

$$\begin{tikzcd}
P_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P_{\mathcal{B}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P_{\mathcal{D}} \arrow[d] & \\
\mathcal{C}_S \arrow[r] \arrow[d, "\pi_S"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{B}} \arrow[r] \arrow[d, "\pi_{\mathcal{B}}"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{D}} \arrow[r] \arrow[d, "\pi_{\mathcal{D}}"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{M}} \arrow[d, "\pi_{\mathcal{M}}"] \\
S \arrow[r, "\pi_{S/\mathcal{B}}"] & \mathcal{B} \arrow[r, "\pi_{\mathcal{B}/\mathcal{D}}"] & \mathcal{D} \arrow[r, "\pi_{\mathcal{D}/\mathcal{M}}"] & \mathcal{M}
\end{tikzcd}$$

$\leftarrow$ universal principal bundle

$\leftarrow$ universal curve

$$\begin{tikzcd}
\mathcal{V}_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{V}_{\mathcal{B}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{V}_{\mathcal{D}} = P_{\mathcal{D}}\times_\rho V \arrow[d] \\
\mathcal{C}_S \arrow[r] \arrow[d, "\pi_S"] \arrow[u, bend left, "u_S"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{B}} \arrow[r] \arrow[d, "\pi_{\mathcal{B}}"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{D}} \arrow[d, "\pi_{\mathcal{D}}"] \\
S \arrow[r, "\pi_{S/\mathcal{B}}"] & \mathcal{B} \arrow[r, "\pi_{\mathcal{B}/\mathcal{M}}"] & \mathcal{D}
\end{tikzcd}$$

universal section

relative perfect obstruction theory

$$E_{S/\mathcal{B}} = (R\pi_{S*}\mathcal{V}_S)^\vee$$