Wednesday, 2/7/2024

Enumerative Geometry of GLSMs II

§3 History and Motivation

- $W_5(x_1,\dots,x_5) = x_1^5+\dots+x_5^5$ Fermat quintic polynomial
- $Q = \{W_5(x_1,\dots,x_5)=0\} \subset \mathbb{P}^4$ quintic CY 3-fold

(or more generally, any smooth deg 5 hypersurface in $\mathbb{P}^4$)

$h^{1,1}(Q) = 1$ $\quad h^{2,1}(Q) = \dim H^1(Q, T_Q) = 101$

- μ_5 = group of 5-th roots of unity $= \langle \zeta \rangle$, $\zeta = e^{2\pi\sqrt{-1}/5}$
- Mirror family $\check{Q}_\psi$: crepant resolution of

$$\{y_1^5+\dots+y_5^5 - 5\psi y_1\cdots y_5 = 0\} \subset \mathbb{P}^4/G_5$$

$$G_5 = \{(a_1\dots a_5) \in (\mu_5)^5 : a_1\cdots a_5 = 1\} / \{(a,\dots,a) : a \in \mu_5\} \cong (\mu_5)^3$$

$h^{1,1}(\check{Q}_\psi) = 101$, $h^{2,1}(\check{Q}_\psi) = 1$

The (compactified) complex moduli of $\check{Q}_\psi$ is $\bar{\mathcal{M}} = \mathbb{P}[5,1]$

obtained by gluing $\mathbb{C}_z = \mathrm{Spec}\,\mathbb{C}[z]$ and

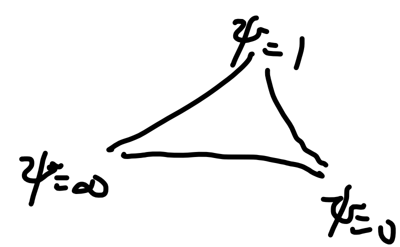
$[\mathbb{C}_\psi/\mu_5] = [\mathrm{Spec}(\mathbb{C}[\psi])/\mu_5]$ via $\mathbb{C}_z^* \to \mathbb{C}_\psi^*/\mu_5$

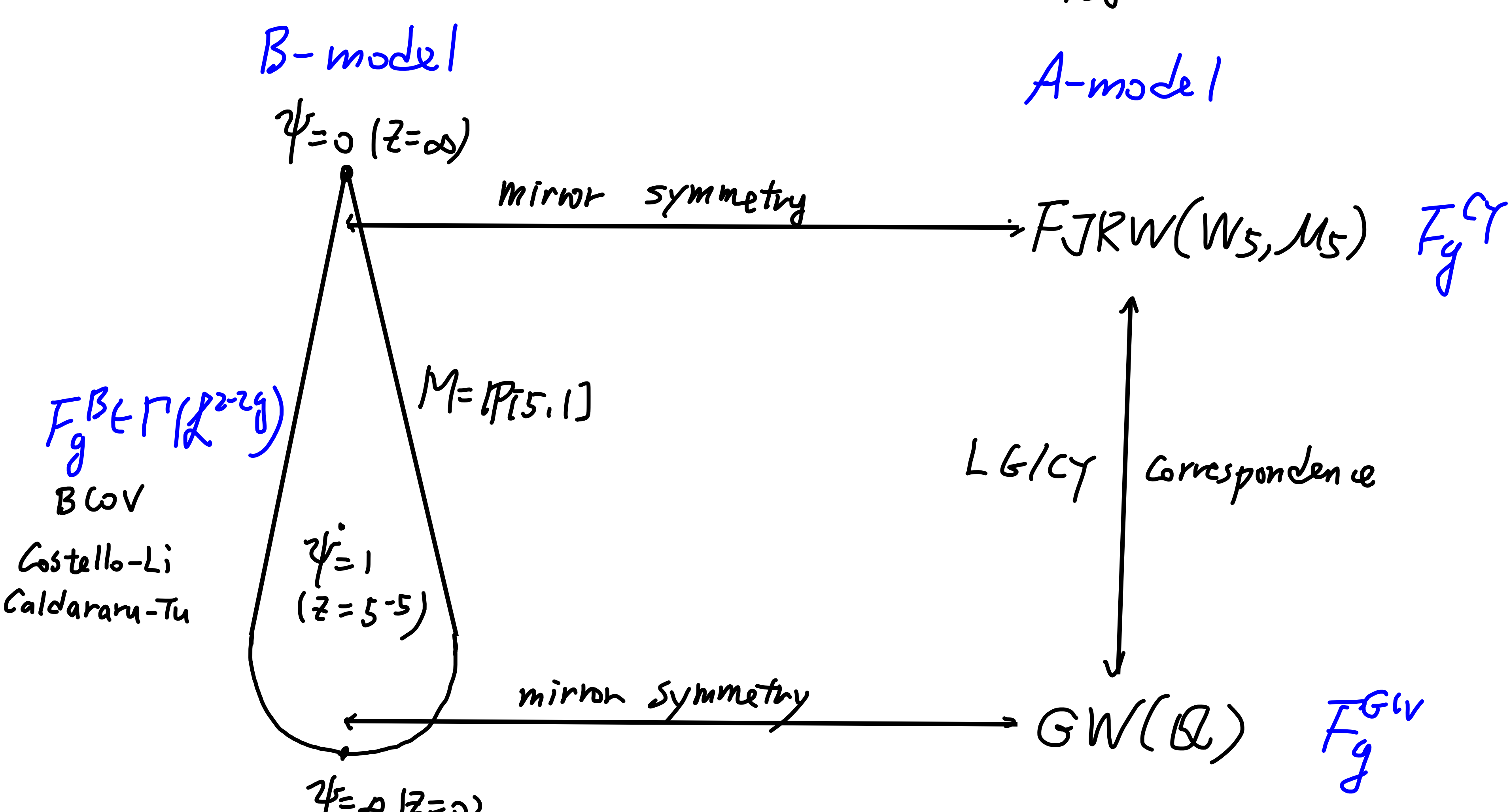
$$z \mapsto (5\psi)^{-5}$$

- $\psi = \infty$ $(z=0)$: MUM (maximal unipotent monodromy) point
- $\psi = 1$ $(z = 5^{-5})$: conifold point
- $\psi = 0$ $(z=\infty)$: orbifold point

Hodge line bundle

$$\begin{array}{ccccc} H^{3,0}(\check{Q}_\psi, \mathbb{C}) = H^0(\check{Q}_\psi, \Omega^3_{\check{Q}_\psi}) & \subset & \mathcal{F}^3 = \mathcal{L}^\vee = \mathcal{L}^{-1} & & \\ \downarrow & & \downarrow & & \\ \psi & \in & \mathcal{M} & \subset & \overline{\mathcal{M}} = \mathbb{P}[5,1] \end{array}$$





$\boxed{g=0}$ Conjectured by Candelas-dela Ossa-Green-Parkes 1991

GW first proved by Givental 1996

Lian-Liu-Yau 1997

FJRW Chiodo-Ruan 2008

genus-0 mirror theorem for FJRW(W_5, μ_5)

genus-0 LG/CY correspondence for quintic 3-fold Q

$\boxed{g=1}$ Conjectured by Bershadsky-Cecotti-Ooguri-Vafa (BCOV) in 1993

GW first proved by A. Zinger (2007)

using genus-one reduced GW theory developed by

LI-Zinger, Vakil-Zinger

LI = Jun Li

Li = Wei-Ping Li

FJRW Guo-Ross

2016 genus-one mirror theorem for FJRW(W_5, μ_5)

2017 genus-one LG/CY correspondence for quintic CY 3-fold Q

using Mixed-Spin-Fields (MSP) theory by Chang-LI-Li-L

$\boxed{g \geq 2}$ BCOV 1993 $g=2$

GW Huang-Klemm-Quackenbush 2006 $g \leq 51$

based on the work of BCOV and

Yamaguchi-Yau
- polynomiality / finite generation
- YY functional equation (holomorphic version of BCOV HAE)

Holomorphic Anomaly Equation

Guo-Janda-Ruan (2017) $g=2$ BCOV conjecture

using log GLSM developed by Chen-Janda-Ruan

all genus polynomiality / finite generation & HAE:

① Chang-Guo-LI (2018) ← using NMSP (Guo-Chang-LI-Li)

② Guo-Janda-Ruan (2018) ← using log GLSM (Chen-Janda-Ruan)

GLSM (Fan-Jarvis-Ruan)

" Both ① and ② are inspired by

E. Witten, "Phases of $N=2$ Theories In Two Dimensions" 1993

2023 GLSM@30

R-symmetries

Last time: input data of GLSM $(V, G, \mathbb{C}^*_R, W, \theta)$

$\mathbb{C}$ vector space — gauge group — superpotential

$\theta \in \hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$ stability condition

today $\theta \in \hat{G}_{\mathbb{R}} := \hat{G} \otimes \mathbb{R}$

$$X_\theta = [V /\!\!/_\theta G] = [\mu^{-1}(\theta)/G_{\mathbb{R}}] \qquad G_{\mathbb{R}} \subset G$$

maximal compact

$\mu : V \to \hat{G}_{\mathbb{R}}$ moment map

Example (GLSM for the quintic CY 3-fold)

$V = \mathrm{Spec}\, \mathbb{C}[x_1, \cdots, x_5, p] = \mathbb{C}^6$

$$\begin{array}{lcccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & p \\ G = \mathbb{C}^* \text{ acts by weights} & (1, & 1, & 1, & 1, & 1, & -5) \\ \mathbb{C}^*_R & (0, & 0, & 0, & 0, & 0, & 1) \end{array}$$

gauge charges

$G \cap \mathbb{C}^*_R = \{1\}$, $\quad \mathbb{C}^*_R = \mathbb{C}^*_W$

$$W = p(x_1^5 + \cdots + x_5^5) = p W_5(x)$$

$$dW = W_5(x)dp + \sum_{i=1}^{5} 5 p x_i^4 dx_i$$

$$\mu: \left(\mathbb{C}^6, \frac{\sqrt{-1}}{2}\left(\sum_{i=1}^{5} dx_i \wedge d\bar{x}_i + dp \wedge d\bar{p}\right)\right) \longrightarrow \mathbb{R}$$

$$\mu(x_1, \cdots x_5, p) = \frac{1}{2}\left(\sum_{i=1}^{5} |x_i|^2 - 5|p|^2\right)$$

$\theta \in \mathbb{R}$ $\quad \mathcal{X}_\theta = [\mu^{-1}(\theta)/U(1)]$ $\quad V^{ss}(\theta) = V^s(\theta) \iff \theta \neq 0$

<u>$\theta > 0$: CY phase</u>

$$\mathcal{X}_\theta = X_\theta = ((\mathbb{C}^5_x - \{0\}) \times \mathbb{C}_p)/G$$

$$= \mathrm{tot}(\mathcal{O}_{\mathbb{P}^4}(-5)) = K_{\mathbb{P}^4}$$

$$\mathcal{Z}_\theta = Z_\theta = Q \underset{W_5=0}{\subset} \mathbb{P}^4 \underset{p=0}{\subset} K_{\mathbb{P}^4}$$

Fermat quintic CY 3-fold

$\{W_5(x_1 \cdots x_5) = 0\} \subset \mathbb{P}^4$

GLSM invariants

$= \pm$ GW invariants of Q

$\Uparrow$

Chang-Li

<u>$\theta < 0$: LG phase</u>

$$\mathcal{X}_\theta = [(\mathbb{C}^5_x \times (\mathbb{C}_p - \{0\}))/G]$$

$$= [(\mathbb{C}^5 \times \{1\})/\mu_5] = [\mathbb{C}^5/\mu_5]$$

$$X_\theta = \mathbb{C}^5/\mu_5$$

$$Z_\theta = \{x_1^4 = 0\} \subset \mathbb{C}^5/\mu_5$$

GLSM invariants

$=$ FJRW invariants of (W_5, μ_5)

Wall-Crossing in GLSM

Chiodo-Ruan (2008) genus-0 LG/CY correspondence

(ε-wall-crossing) Givental style mirror theorems

- CY phase (Givental, LLY) $J^+ = \dfrac{I^+}{I_0^+}$ under the mirror map

- LG phase (Chiodo-Ruan) $J^- = \dfrac{J^-}{I_0^-}$ under the mirror map

$I^\pm, J^\pm$ are functions in one-variable

take values in a 4-dim'l complex symplectic space

$$H(z)_\pm = z H_\pm^0 \oplus H_\pm^2 \oplus z^{-1} H_\pm^4 \oplus z^{-2} H_\pm^6$$

$H_+^{2i} = \mathbb{C} H^i$ $\quad$ H hyperplane class

(θ-wall-crossing / ζ-wall-crossing)

$I^+ = I^{CY}$ and $I^- = I^{LG}$ are related by change of variable

analytic continuation, and $z = (5^5 t)^{-5} = t^{-5}$

a $\mathbb{C}$-linear symplectic isomorphism $\underset{\in Sp(4,\mathbb{C})}{\phi} : H(z)_+ \to H(z)_-$