Thursday, 2/8/2024

Enumerative Geometry for GLSMs III

Recall:

Monday (2/5)

§1 GLSM input data $\underline{\mathfrak{X}} = (V, G, \mathbb{C}^*_R, W, \theta)$

$V = \operatorname{Spec}\mathbb{C}[x_1,\dots,x_N] = \mathbb{C}^N$

$$G \subset GL(V) \qquad G \cap \mathbb{C}^*_R = \langle J \rangle = \mu_r$$

$$\cup$$

Commute $\mathbb{C}^*_R$

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{C}^*_\omega \to 1$$

$$\Gamma = G\mathbb{C}^*_R, \qquad \mathbb{C}^*_\omega = \Gamma/G$$

$W \in \mathbb{C}[x_1,\dots,x_N]^G$

$W(t\cdot x) = t^r W(x) \qquad t \in \mathbb{C}^*_R$

§2 prestable LG quasimaps to $\underline{\mathfrak{X}}$

$$\underline{u} = ((C, \underbrace{z_1,\dots,z_\ell}_{\underline{z}}), P, \kappa, u)$$

twisted curve

Γ-bundle

$\kappa : P \times_\chi \mathbb{C} \xrightarrow{\cong} \omega_C^{\log} = \omega_C(z_1 + \cdots + z_\ell)$

$u \in H^0(P \times_\rho V)$

$$
\begin{tikzcd}
LG(\underline{x}) \arrow[r, phantom, "\overset{\text{open}}{\subset}"] & \mathcal{S} \arrow[d, "\text{virtually smooth}"] & ((\mathcal{C}, \underline{z}), \mathcal{P}, \mathcal{K}, u) \\
& \mathcal{B} \arrow[d] & ((\mathcal{C}, \underline{z}), \mathcal{P}, \mathcal{K}) \\
& \mathcal{D} \arrow[d] & ((\mathcal{C}, \underline{z}), \mathcal{P}) \\
& \mathcal{M} & (\mathcal{C}, \underline{z})
\end{tikzcd}
$$

smooth Artin

$B(u) := \mathcal{P} \times_{\rho} V_G^{un}(\theta)$

finite, disjoint from nodes and markings

Wednesday (2/7)

§3 History and motivation

§4 Stability and properness

$$\theta \in \hat{G}_{\mathbb{Q}} := \hat{G} \otimes_{\mathbb{Z}} \mathbb{Q}$$

Definition

$\vartheta \in \hat{\Gamma}_{\mathbb{Q}}$ is a lift of $\theta \in \hat{G}_{\mathbb{Q}}$ if $\vartheta|_G = \theta$ always exist

$\left(\Rightarrow V_\Gamma^{ss}(\vartheta) \subset V_G^{ss}(\vartheta)\right)$ does not always exist

We say it is a good lift if $V_\Gamma^{ss}(\vartheta) = V_G^{ss}(\theta)$

<u>Exercise</u> (quintic)

$V = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_5, p] = \mathbb{C}^6$

$G = \mathbb{C}^* \quad (1,1,1,1,1,-5)$

$\mathbb{C}^*_R \quad (0,0,0,0,0,1) \qquad W = p \sum_{i=1}^{5} x_i^5$

$\exists$ good lift $\forall \theta \in \hat{G}_{\mathbb{Q}} \cong \mathbb{Q}$

<u>Exercise</u> (master space)

$V = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4, x_5, p, u, v] = \mathbb{C}^8$

$(s_1, s_2) \in G = (\mathbb{C}^*)^2 \quad (1,1,1,1,1,-5,1,0)$

$\qquad (0,0,0,0,0,0,1,1)$

$\mathbb{C}^*_R \quad (0,0,0,0,0,1,0,0) \qquad W = p \sum_{i=1}^{5} x_i^5$

Let $\theta(s_1, s_2) = s_1 s_2^2$

Find $V_G^{ss}(\theta)$, $\mathcal{X}_\theta = [V /\!\!/_\theta G] = [V_G^{ss}(\theta) / G]$

Show that there does not exist a good lift

Find $Z(dW)$, $\mathcal{Z}_\theta = [Z(dW) /\!\!/_\theta G]$

Definition Let $\vartheta \in \hat{\Gamma}_{\mathbb{Q}}$ be a good lift of $\theta \in \hat{G}_{\mathbb{Q}}$

$\underline{u} = ((C, \underline{z}), P, \kappa, u)$ prestable LG quasimap to

$\underline{\mathcal{X}} = (V, G, \mathbb{C}_R^*, W, \theta)$ or more generally

$\underline{\mathcal{Y}} = (Y, G, \mathbb{C}_R^*, W, \theta)$ Y affine Γ subvariety of V

$u \in \Gamma(P \times_\Gamma Y)$

The length of y in C w.r.t. u and ϑ

$$\ell^{u,\vartheta}(y) := \min\left\{ \frac{(u^* s)_y}{m} \;\middle|\; s \in H^0(Y, L_\theta^m)^\Gamma, m \in \mathbb{Z}_{>0} \right\} \in \mathbb{Z}_{\geq 0}$$

Remarks • $\ell^{u,\vartheta}(y) > 0 \iff y \in Bl_u$

• $k \in \mathbb{Z}_{>0}$, $\vartheta^{u,k\vartheta}(y) = k\vartheta^{u,\vartheta}(y)$

Definition A prestable LG quasimap $\underline{u}$ to $\underline{\mathcal{X}}$ is (ϑ, ε) stable (where $\varepsilon \in \mathbb{Q}_{>0}$) if

(1) $\omega_C^{\log} \otimes (P \times_\vartheta \mathbb{C})^\varepsilon$ is an ample $\mathbb{Q}$-line bundle

(2) $\varepsilon\, \ell^{u,\vartheta}(y) \leq 1$ $\forall y$ in C

Remark

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\text{normalization}} & C \\ \cup & & \cup \\ \text{connected component } \tilde{C}_v & \longrightarrow & C_v \ \text{irreducible component} \end{array}$$

(1) $\Longleftrightarrow$ $\forall$ connected component $\tilde{C}_v$ of $\tilde{C}$

$$2g_v - 2 + l_v + \varepsilon \underbrace{\deg\left((P^{\times}_{\vartheta}\mathbb{C})|_{e_v}\right)}_{\geq 0} > 0$$

$g_v = \text{genus}(\tilde{C}_v)$ $\quad$ l_v = number of special points in $\tilde{C}_v$

mapped to nodes or marked points

$2g_v - 2 + l_v > 0$ ✓

$\varepsilon \to 0^+$ $\quad$ $(g_v, l_v) \neq (0,1)$ $\quad$ no rational tail

$\quad (g_v, l_v) = (0,2) \Rightarrow \deg\left((P^{\times}_{\vartheta}\mathbb{C})|_{e_v}\right) > 0$ $\quad$ rational bridge

$\varepsilon \to +\infty$ $\quad$ $(g_v, l_v) = (0,1), (0,2) \Rightarrow \deg\left((P^{\times}_{\vartheta}\mathbb{C})|_{e_v}\right) > 0$

(2) $\varepsilon \to +\infty$ $\quad$ $\ell^{u,\vartheta}(y) = 0$ $\quad \forall y$ in $C \Rightarrow B(u)$ is empty

$u: C \to [V^{ss}_{\rho}(\vartheta)/\Gamma]$

$$
\begin{tikzcd}
 & {[V/\Gamma]} \arrow[d] \\
C \arrow[ur, "{[u]}"] \arrow[r, "P"] \arrow[dr, "\omega_C^{\log}"'] & B\Gamma = [\bullet/\Gamma] \arrow[d, "B\chi"] \\
 & B\mathbb{C}^*_\omega
\end{tikzcd}
$$

$$
\begin{aligned}
\deg([u]) \in H_2([V/\Gamma];\mathbb{Q}) &= H_2(B\Gamma;\mathbb{Q}) \\
&= \underbrace{H_2(BG;\mathbb{Q})}_{=\operatorname{Hom}(\hat{G},\mathbb{Q}) \,\ni\, d} \oplus \underbrace{H_2(B\mathbb{C}^*_\omega;\mathbb{Q})}_{=\mathbb{Q} \,\ni\, 2g-2+\ell}
\end{aligned}
$$

$LG^{\vartheta,\varepsilon}_{g,\ell}(\mathcal{Y},d)$ moduli of genus g, ℓ-pointed, deg d ε-stable quasi-stable LG maps to $\mathcal{Y}$

<u>Theorem</u> (FJR = Fan-Jarvis-Ruan)

$LG^{\varepsilon,\vartheta}_{g,\ell}(\mathcal{Y},d)$ is a separated DM stack of finite type. It is proper over $\bullet = \operatorname{Spec}\mathbb{C}$ if $\mathcal{Y}_0 = [(Y \cap V^{ss}_G(\theta))/G]$ is.

CGLLZ = Chang-Guo-LI-Li-Zhou December 2023

Ω-stability applicable to the master space

<u>Theorem</u> (CGLLZ)

$LG^{\Omega}_{g,\ell}(\mathcal{Y},d)$ is a separated DM stack of finite type. It is proper over $\bullet = \operatorname{Spec}\mathbb{C}$ if $\mathcal{Y}_0$ is.

§ 5 Inertia stacks and evaluation maps

$\Psi \in \mathrm{Conj}(G)$

$I(\Psi) = \{ (v, r) \in V_G^{ss}(\theta) \times \Psi : \underset{\rho(r)v}{\underset{\shortparallel}{r \cdot v}} = v \}$ $\qquad \rho : \Gamma \to GL(V)$

$G \circlearrowright \quad a \cdot (v, r) = (a \cdot v, a r a^{-1})$

$\mathcal{X}_{\theta, \Psi} = [I(\Psi)/G]$

$B := \{ \Psi \in \mathrm{Conj}(G) : I(\Psi) \neq \phi \}$ is finite

The initia stack of $\mathcal{X}_\theta$ is

$I\mathcal{X}_\theta = \bigsqcup_{\Psi \cup B} \mathcal{X}_{\theta, \Psi}$ disjoint union of connected components

$\mathcal{X}_{\theta, \{1\}} \subseteq \mathcal{X}_\theta$

ω_e^{Log} is trivialized at marked points

$ev_i : LG_{g,l}(\underline{x}, d) \longrightarrow I\mathcal{X}_\theta = \bigsqcup_{\Psi \in B} \mathcal{X}_{\theta, \Psi}$

$LG_{g, \Psi_1, \ldots, \Psi_l}(\underline{x}, d) = \bigcap_{i=1}^{l} ev_i^{-1}(\mathcal{X}_{\theta, \Psi})$

§6 Virtual matrix factorization and virtual fundamental classes

Favero-Kim : generalize Polishchuk-Vaintrob (G finite abelian)

and CF-Favero-Kim-Guéré-Shoemaker (convex hybrid model)

CF = Ciocan-Fontanine

$$\begin{array}{ll}
\mathcal{V}_B = P_{\mathcal{B}} \times_\rho V & \\
\downarrow & \\
\mathcal{C}_{\mathcal{B}} & \\
\downarrow \pi_{\mathcal{B}} & \\
\mathcal{B} & \{((C, \underline{z}), P, \kappa)\}
\end{array}$$

$\hookrightarrow$ universal moduli of Γ-structures

$$\mathcal{S} = C(\pi_* \mathcal{V}_{\mathcal{B}}) := \mathrm{Spec}_{\mathcal{B}}(\mathrm{Sym}^\bullet R^1\pi_*(\mathcal{V}_{\mathcal{B}}^\vee \otimes \omega_{\pi_{\mathcal{B}}}))$$

fiber $H^0(C, P \times_\rho V)$

$$\begin{array}{ccccc}
X = LG_{g,\underline{\Psi}}(\underline{x}, d) & \subset & \mathcal{S}_{g,\underline{\Psi}}(\underline{x}, d) \\
\downarrow & & \downarrow \\
\mathcal{B}_0 & \overset{\text{open}}{\subset} & \mathcal{B}
\end{array}$$

$\underline{\Psi} = (\Psi_1, \ldots, \Psi_\ell)$

$$E^\vee_{\mathcal{S}/\mathcal{B}} = \pi^*_{\mathcal{S}/\mathcal{B}} R\pi_{\mathcal{B}*} \mathcal{V}_{\mathcal{B}}$$

finite type

$$\begin{tikzcd} X \subset C \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ \mathcal{B}^\circ \arrow[r, hook] & \mathcal{B} \end{tikzcd}$$

$$ev_i^C : C \to \mathcal{X}_{\Phi_i} \subset [V/G]$$

$$ev_i : X \to \mathcal{X}_{\theta,\Phi_i} \subset [V_G^{ss}(\theta)/G]$$

Over the finite type smooth Artin stack $\mathcal{B}^\circ$

$R\pi_{\mathcal{B}^\circ *}\mathcal{V}_{\mathcal{B}^\circ}$ admits a global resolution

$$R\pi_{\mathcal{B}^\circ *}\mathcal{V}_{\mathcal{B}^\circ} = [\underline{A} \xrightarrow{d_A} \underline{B}]$$

locally free sheaf of $\mathcal{O}_{\mathcal{B}^\circ}$ modules

$$\begin{tikzcd} \pi_{\mathcal{B}^\circ *}\mathcal{V}_{\mathcal{B}^\circ} \arrow[r] \arrow[dr] & \underline{A} \arrow[r, "d_A"] \arrow[d] & \underline{B} \arrow[dl] \\ & \mathcal{B}^\circ & \end{tikzcd}$$

Taking C

$$\begin{tikzcd} \pi_{A/B}^* \underline{B} \arrow[d] \\ C = Z(\beta_A) = A = \mathrm{tot}(\underline{A}) \arrow[u, bend left, "\beta_A = \pi_{A/B}^* d_A \circ t_A"'] \arrow[r, "\pi_{A/B}"] & B = \mathrm{tot}(\underline{B}) \end{tikzcd}$$

$$\begin{tikzcd}
X = Z(\beta_U) \arrow[r, hook, "i_X"] \arrow[d, "j_X"] & U \arrow[r, "ev_i"] \arrow[d, "j_U"] & \mathcal{X}_{\theta,\Psi_i} \arrow[d] \\
C = Z(\beta_A) \arrow[r, hook, "i_C"] & A \arrow[r] & \mathcal{X}_{\Psi_i}
\end{tikzcd}$$

$\to$ smooth DM

smooth Artin

$$W_U = \sum_{i=1}^{\ell} (ev_i^U)^* W_{\theta,\Psi_i} = j_U^* W_A$$

$$W_A = \sum_{i=1}^{\ell} (ev_i^A)^* W_{\Psi_i}$$

We now describe Favero-Kim construction in the following nice case:

(a) U can be chosen to be separated

(b) $\exists$ cosection $\alpha_A^\vee : \pi^*_{A/B^\circ} \underline{B} \to \mathcal{O}_A \Leftrightarrow \alpha_A : \mathcal{O}_A \to \pi^*_{A/B^\circ} \underline{B}^\vee$

$$\langle \alpha_A, \beta_A \rangle = -W_A$$

(c) Let $\alpha_U = j_U^* \alpha_A \in \Gamma(U, \underline{B}_U^\vee)$

$$\begin{array}{ccccc}
Z = Z(\alpha_U) \cap Z(\beta_U) & \subset & X = Z(\beta_U) & \subset & U \\
\| & & \| & & \\
LG_{g,\Psi}(\underline{\mathcal{Z}}, d) & & LG_{g,\Psi}(\underline{\mathcal{X}}, d) & &
\end{array}$$

When α_A exists

$$\{\alpha_A, \beta_A\} = \left[\bigoplus_i \Lambda^{2i} \pi^*_{A/B^\circ} \underline{B}^\vee \underset{\partial}{\overset{\partial}{\rightleftarrows}} \Lambda^{2i+1} \pi^*_{A/B^\circ} \underline{B}^\vee \right]$$

$$\partial = i_{\beta_A} + \alpha_A \wedge$$

virtual matrix factorization

$$\mathbb{K}_U := \{\alpha_U, \beta_U\} = j_U^* \{\alpha_A, \beta_A\}$$

Koszul matrix factorization of $(U, -W_U)$

virtual fundamental class

$$[U]^{vir}_W := \left(\prod_{i=1}^{l} r_i\right) td(\underline{B}_U)\, ch_Z^U \mathbb{K}_U \in \mathbb{H}^{even}_Z(U, (\Omega_U^\bullet, -dW_U \wedge))$$

$r_i = ord(r)$ $\quad r \in \Gamma_i$

In the general case α_A only exists locally on U

$$\alpha_A \in Th^\bullet G^\bullet K(\beta_A) \qquad K(\beta_A) = Sym^{rank \underline{B}}(\pi_{A/B}^* \underline{B}^\vee \xrightarrow{i_\beta} \mathcal{O}_U)$$

$$d(\alpha_A) = -W_A$$

$$\mathbb{K}_U = [\, Th^\bullet G^\bullet K(\beta_U)^{even} \underset{\partial}{\overset{\partial}{\rightleftarrows}} Th^\bullet G^\bullet K(\beta_U)^{odd} \,]$$

$$\partial = \iota_{\beta_U} + d_{dR} + \alpha_U \wedge$$

matrix factorization for $(U, -W_U)$

locally a Kozul matrix factorization

$$[U]^{vir}_W = \left(\prod_{i=1}^{l} r_i\right) td\, ch_Z^U \mathbb{K}_U \in \mathbb{H}^{even}_Z(U, (\Omega_U^\bullet, -dW_U \wedge))$$

$$\begin{tikzcd}
X \subset U \arrow[r] \arrow[rrr, bend left, "\mathrm{forget}^U"] & \mathcal{B} \arrow[r] & \mathcal{M}^{tw}_{g,\ell} \arrow[r] & \overline{M}_{g,\ell}
\end{tikzcd}$$
$$X \xrightarrow{\mathrm{forget}} \overline{M}_{g,\ell}$$

Favero-Kim

$$(ev^U, \mathrm{forget}^U)_* [U]^{vir}_W \in \mathbb{H}^*(I\mathcal{X}_\theta, (\Omega^\bullet_{I\mathcal{X}_\theta}, -dW_\theta \wedge))^{\otimes \ell} \otimes H^*(\overline{M}_{g,\ell})$$

defines CohFT

$$\Omega_{g,\ell,d} : \mathbb{H}(I\mathcal{X}_\theta, (\Omega^\bullet_{I\mathcal{X}_\theta}, dW_\theta \wedge))^{\otimes \ell} \longrightarrow H^*(\overline{M}_{g,\ell})$$

<u>unit</u> $\mathbb{C}^*_R$ on V by weights $c_1, \dots c_N \in \mathbb{Z}$

$$G \cap \mathbb{C}^*_R = \langle J \rangle \cong \mu_r, \text{ where } J = (e^{2\pi\sqrt{-1}c_1/r}, \dots, e^{2\pi\sqrt{-1}c_N/r})$$

Assume $r \geq c_i \geq 0$, $\mathcal{X}_\theta^{\mathbb{C}^*_R} = [(V^{ss}_G(\theta) \cap V^{\mathbb{C}^*_R})/G]$

$$\mathcal{X}_\theta^{\mathbb{C}^*_R} = Z(\beta) \hookrightarrow \mathcal{X}_{\theta,\{J\}}$$

$$\langle \alpha, \beta \rangle = W_{\theta,\{J\}} := W_\theta|_{\mathcal{X}_{\theta,\{J\}}}$$

$\{\alpha, \beta\}$ Koszul factorization of $(\mathcal{X}_{\theta,\{J\}}, W_{\theta,\{J\}})$

unit $\mathbb{1} := \mathrm{td}\,\mathrm{ch}\{\alpha, \beta\} \in \mathbb{H}(\mathcal{X}_{\theta,\{J\}}, (\Omega^\bullet_{\mathcal{X}_{\theta,\{J\}}}, dW_{\theta,\{J\}} \wedge))$