F-regularity and finite generation

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Throughout, ①

$k :=$ perfect field of char. $p > 0$;
$K :=$ finitely generated extension of k;
$R :=$ k-subalgebra of K with

$$\mathrm{Frac}(R) = K.$$

(not assuming R is noeth.)

<u>Def^n</u> R is <u>split F-regular</u> (SFR) if

$$\forall\ r \in R - \{0\},$$

$\exists\ e \in \mathbb{Z}_{>0}$ and an R-linear map

$$\begin{aligned} R^{1/p^e} &\longrightarrow R. \\ r^{1/p^e} &\longmapsto 1 \end{aligned}$$

When R is noetherian this notion is called STRONGLY F-REGULAR [Hochster-Huneke]

R is SFR

$\Downarrow$

R is a summand of finite extensions (SPLINTER)

$\Downarrow$

R is normal

EXAMPLES

HOCHSTER, HUNEKE, BENITO, MULLER, RAJCHGOT, SMITH

- Coordinate ring of an AFFINE NORMAL toric k-variety.
- RING OF INVARIANTS of a finite group G acting on $k[x_1, \ldots, x_n]$ such that $p \nmid |G|$. (More general versions due to Hashimoto)
- GENERIC DETERMINANTAL rings.
- LOCALLY ACYCLIC cluster algebras.
- R is EXCELLENT REGULAR.

CRITICISM All examples are noeth.

CONJECTURE [SINGH, SCHWEDE]

R is SFR $\Longrightarrow$ R is noeth.

CONSEQUENCES IF CONJECTURE HOLDS

1) [SMITH, HASHIMOTO]

X is globally F-regular +

D is an effective Cartier divisor on X

$$\Downarrow$$

$$\bigoplus_{n \in \mathbb{N}} \Gamma(X, nD)$$

is SFR.

2) [WATANABE, DE STEFANI - MONTAÑO - NÚÑEZ-BETANCOURT]

$(R, \mathfrak{m})$ is noeth. local, SFR and I is a pure ht 1 ideal (i.e. divisorial ideal)

$$\Downarrow$$

$$\bigoplus_{n \in \mathbb{N}} I^{(n)} \text{ is SFR}$$

Upshot Rings in 1) + 2) are noetherian if conjecture holds.

3) [ABERBACH – HUNEKE – POLSTRA]
F-regular $\Longrightarrow$ strongly F-regular
follows from 2).

EVIDENCE FOR CONJECTURE

1) [HACON-XU, BIRKAR, HARA-WATANABE, SCHWEDE-SMITH]
Divisorial symb. Rees-algs. are noeth. if
- R is SFR + e.f.t $/k$,
- $\dim(R) \leq 3$ and
- $p > 5$.

2) [D-SMITH] R is a SFR valuation ring of K/k $\Rightarrow$ R is eft $/k$.

MAIN THEOREM [D-SCHWEDE-TUCKER]

$\sigma \subseteq \mathbb{R}^d$ is a full dim'l CONVEX CONE; $0 \in \sigma$.

The monoid alg. $k[\sigma \cap \mathbb{Z}^d]$ is SFR

$\Downarrow$

1) $\sigma = \overline{\sigma}$,

2) σ is rational polyhedral.

Gordan's Lemma $\Rightarrow k[\sigma \cap \mathbb{Z}^d]$ is a f.g. k-alg.

Carathéodory's Thm + normality of SFR rings

$$\Downarrow$$

COROLLARY [DST]

Let M be a sub-monoid of $\mathbb{Z}^d$.
$k[M]$ SFR $\Rightarrow M$ is a f.g. monoid

BACKGROUND FROM CONVEX GEOMETRY

- $\sigma \subseteq \mathbb{R}^d$ is a CONVEX CONE if
  $\forall\, x, y \in \sigma,\ s, t \in \mathbb{R}_{\geq 0},$
  $$sx + ty \in \sigma.$$
  $$\therefore\ 0 \in \sigma$$

- The DUAL of σ is
  $$\sigma^\vee := \left\{ \varphi : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } \varphi(\sigma) \subseteq [0, \infty) \right\}$$
  - $\sigma^\vee$ is a closed convex cone in $(\mathbb{R}^d)^\vee$.
  - $(\sigma^\vee)^\vee = \overline{\sigma}$.

- The LINEALITY SPACE $L(\sigma)$ of σ is the LARGEST linear space in σ.
  $$L(\sigma) = \sigma \cap -\sigma.$$

- σ is POINTED if $L(\sigma) = \{0\}$.

$$\pi : \mathbb{R}^d \xrightarrow[\text{proj.}]{} \mathbb{R}^d / L(\sigma)$$

- $\pi(\sigma)$ is pointed ;
- $\sigma \underset{\substack{\uparrow \\ \text{not canonical}}}{\simeq} L(\sigma) \times \pi(\sigma)$

- A ray τ of σ is EXTREMAL if $\forall x, y \in \sigma$, $x+y \in \tau \Rightarrow x, y \in \tau$.

σ has extremal rays $\Rightarrow$ σ is pointed

$\boxed{\text{WARNING}}$ $\{\text{extremal rays}\}$
$\not\subset \{\sigma \cap \varphi^{\perp} : \varphi \in \sigma^{\vee}\}$.

- CLOSED POINTED CONES are generated by their extremal rays.

- A ray τ of σ is RATIONAL if $\exists$ a NONZERO

$$x \in \tau \cap \mathbb{Z}^d.$$

Then $\tau = \mathbb{R}_{\geq 0} \cdot x$.

- The RELATIVE INTERIOR of σ

$$\mathrm{Relint}(\sigma)$$

is the topological interior of σ in

$$\mathrm{Span}(\sigma).$$

$\mathrm{Relint}(\sigma) = \mathrm{Int}(\sigma)$ if σ is full dim'l.

- $\forall\, x \in \mathrm{Relint}(\sigma),\ y \in \bar{\sigma},$

$$x+y \in \mathrm{Relint}(\sigma).$$

- σ is full dim'l in $\mathbb{R}^d$ $\Rightarrow$

$$\mathbb{Z}(\sigma \cap \mathbb{Z}^d) = \mathbb{Z}^d$$

$$\{\alpha - \beta \mid \alpha, \beta \in \sigma \cap \mathbb{Z}^d\}$$

NOTATION $\alpha \in \sigma \cap \mathbb{Z}^d \rightsquigarrow x^\alpha \in k[\sigma \cap \mathbb{Z}^d]$.

Observations σ = full dim'l cone in $\mathbb{R}^d$.

a) Any

$$k[\sigma \cap \mathbb{Z}^d]^{1/p^e} \longrightarrow k[\sigma \cap \mathbb{Z}^d]$$

extends to

$$k[\mathbb{Z}^d]^{1/p^e} \longrightarrow k[\mathbb{Z}^d].$$

$k[\mathbb{Z}^d]$ is a localization of $k[\sigma \cap \mathbb{Z}^d]$.

b) $\forall e \in \mathbb{Z}_{>0}$, $k[\mathbb{Z}^d]$ has a CANONICAL Frobenius splitting.

$$\lambda_{0,e} : k[\mathbb{Z}^d]^{1/p^e} \longrightarrow k[\mathbb{Z}^d]$$

$$(X^\alpha)^{1/p^e} \mapsto \begin{cases} X^{\alpha/p^e} & \text{if } \alpha \in p^e \mathbb{Z}^d \\ 0 & \text{otherwise.} \end{cases}$$

Notation / Definition $\forall e \in \mathbb{Z}_{>0}$, $\alpha \in \sigma \cap \mathbb{Z}^d$

define

$$\mathbb{Z}^d_{\alpha,e} := \{ \beta \in \mathbb{Z}^d \mid p^e \beta + \alpha \in \sigma \}$$

Example $\mathbb{Z}^d_{0,e} = \{\beta \in \mathbb{Z}^d \mid p^e\beta \in \sigma\}$

$= \sigma \cap \mathbb{Z}^d.$

c) $\forall\, \alpha \in \sigma \cap \mathbb{Z}^d, \ \exists$

$$\lambda_{\alpha,e} : k[\sigma \cap \mathbb{Z}^d]^{1/p^e} \longrightarrow k[\sigma \cap \mathbb{Z}^d]$$

$$(X^\alpha)^{1/p^e} \longmapsto 1$$

IF AND ONLY IF

$$\mathbb{Z}^d_{\alpha,e} \subseteq \sigma.$$

(depends only on σ, $\mathbb{Z}^d$, e)

$$\boxed{\text{Recall, } \mathbb{Z}^d_{\alpha,e} = \{\beta \in \mathbb{Z}^d \mid p^e\beta + \alpha \in \sigma\}}$$ 16

$\boxed{\Longrightarrow}$ Extend $\lambda_{\alpha,e}$ to

$$\tilde{\lambda}_{\alpha,e} : k[\mathbb{Z}^d]^{1/p^e} \longrightarrow k[\mathbb{Z}^d].$$

For $\beta \in \mathbb{Z}^d_{\alpha,e}$

$$x^\beta = x^\beta \, \tilde{\lambda}_{\alpha,e}\left((x^\alpha)^{1/p^e}\right)$$

$$= \tilde{\lambda}_{\alpha,e}\left((x^{p^e\beta+\alpha})^{1/p^e}\right)$$

$$\underset{p^e\beta+\alpha \in \sigma \cap \mathbb{Z}^d}{=} \lambda_{\alpha,e}\left((x^{p^e\beta+\alpha})^{1/p^e}\right)$$

$$\in k[\sigma \cap \mathbb{Z}^d]$$

$\therefore \beta \in \sigma \cap \mathbb{Z}^d.$

$$\boxed{\text{Recall, } \mathbb{Z}^d_{\alpha,e} = \{\beta \in \mathbb{Z}^d \mid p^e\beta + \alpha \in \sigma\}}$$ 17

$\boxed{\Leftarrow}$

$$k[\mathbb{Z}^d]^{1/p^e} \xrightarrow{(X^{-\alpha})^{1/p^e}} k[\mathbb{Z}^d]^{1/p^e} \xrightarrow{\lambda_{0,e}} k[\mathbb{Z}^d]$$

sends

$$(X^r)^{1/p^e} \mapsto \begin{cases} X^{\frac{r-\alpha}{p^e}} & \text{if } r-\alpha \in p^e\mathbb{Z}^d \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{Z}^d_{\alpha,e} \subseteq \sigma \Rightarrow$ the above map restricted to

$$k[\sigma \cap \mathbb{Z}^d]^{1/p^e}$$

maps into $k[\sigma \cap \mathbb{Z}^d]$.

<u>Example</u> We have seen

$$\mathbb{Z}^d_{0,e} = \sigma \cap \mathbb{Z}^d.$$

$\therefore$ The canonical F-splitting of

$$k[\mathbb{Z}^d]$$

restricts to an F-splitting of

$$k[\sigma \cap \mathbb{Z}^d]$$

for ANY σ.

d) Suppose $\exists\ \alpha \in \operatorname{Relint}(\sigma) \cap \mathbb{Z}^d$ and

$$k[\sigma \cap \mathbb{Z}^d]^{1/p^e} \longrightarrow k[\sigma \cap \mathbb{Z}^d].$$
$$(x^\alpha)^{1/p^e} \longmapsto 1$$

Then $\sigma \cap \mathbb{Z}^d = \bar{\sigma} \cap \mathbb{Z}^d$.

$$\boxed{\text{Recall, } \mathbb{Z}^d_{\alpha,e} = \{\beta \in \mathbb{Z}^d \mid p^e\beta + \alpha \in \sigma\}}$$ 19

Upshot 1) $k[\sigma \cap \mathbb{Z}^d] = k[\bar{\sigma} \cap \mathbb{Z}^d]$

2) $\bar{\sigma}$ generated by $\bar{\sigma} \cap \mathbb{Z}^d$

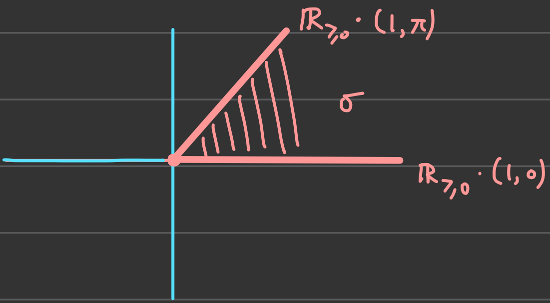
$\Rightarrow \sigma = \bar{\sigma}$.

Proof of d) $\beta \in \bar{\sigma} \cap \mathbb{Z}^d$

$\Rightarrow p^e\beta + \alpha \in \text{Relint}(\sigma)$

$\Rightarrow \beta \in \mathbb{Z}^d_{\alpha,e} \underset{c)}{\subseteq} \sigma \cap \mathbb{Z}^d$.

Low dimensional illustration of MAIN THM



$$k[\sigma \cap \mathbb{Z}^2] = k[\, x^a y^b \mid a, b \in \mathbb{N},\ b \leq \pi a \,]$$

Assume for CONTRADICTION that

$$k[\sigma \cap \mathbb{Z}^2]$$

is split F- regular.

$(1,1) \in \text{Relint}(\sigma) \cap \mathbb{Z}^2.$

Contradiction We'll show $\forall\, e \in \mathbb{Z}_{>0}$,

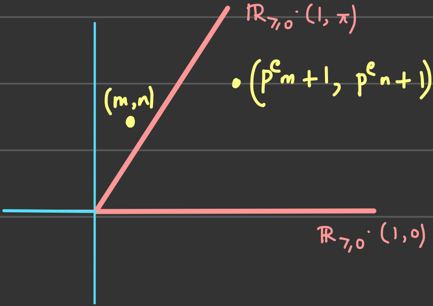
$$\mathbb{Z}^2_{(1,1),e} \not\subseteq \sigma.$$

$$\|$$

$$\{(m,n) \in \mathbb{Z}^2 \mid (p^e m+1,\, p^e n+1) \in \sigma\}$$

WANT $(m,n) \in \mathbb{Z}^2 - \sigma$ s.t.

$$(p^e m+1,\, p^e n+1) \in \sigma.$$



WANT

1) $0 \leq p^e m + 1$

$\iff \quad 0 \leq m$

2) $0 \leq p^e n + 1 \leq \pi (p^e m + 1)$

$\iff \quad 0 \leq n, \quad n - \pi m \leq \dfrac{\pi - 1}{p^e}$

3) $\pi m < n$

2) + 3) $\rightsquigarrow \quad 0 < n - \pi m \leq \dfrac{\pi - 1}{p^e}$.

Pigeonhole P. $\Rightarrow \{\{\pi m\} : m \in \mathbb{Z}_{>0}\}$ is dense in $[0,1]$.

Choose $m \in \mathbb{Z}_{>0}$ s.t.

$$1 - \{\pi m\} \leq \frac{\pi - 1}{p^e},$$

i.e. $$1 + \lfloor \pi m \rfloor - \pi m \leq \frac{\pi - 1}{p^e}.$$

Taking $n = 1 + \lfloor \pi m \rfloor$ we win!

Key tools for MAIN THM in higher dim

1) Diophantine approximation from [BIRKAR-CASCINI-HACON-MCKERNAN '10]
Higher dim'l analog of previous density result.

$k[\sigma \cap \mathbb{Z}^d]$ is SFR $\Rightarrow$

a) $\bar{\sigma} \cap -\bar{\sigma}$ is rat'l

b) $\bar{\sigma}$ is pointed $\Rightarrow$ its extremal rays are rat'l.

c) $\sigma = \bar{\sigma}$ and is gen. by $\sigma \cap \mathbb{Z}^d$.

UPSHOT Reduce to the case of a

- closed, full dim'l, rational, pointed convex cone

and show that its extremal rays DO NOT accumulate.

For this we need:

2) $k[\sigma \cap \mathbb{Z}^d]$ SFR $\Rightarrow$ $\sigma^\vee$ is rat'l.

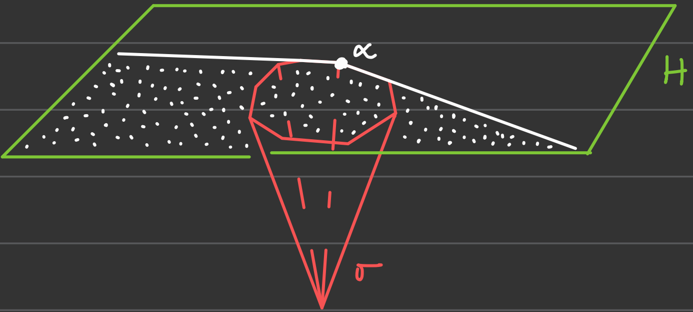
3) Induction on dimension :

H = RATIONAL AFFINE hyperplane of $\mathbb{R}^d$;

$\alpha \in \sigma \cap H \cap \mathbb{Z}^d$;

$H_\alpha = H - \alpha$ (honest hyperplane)

σ_{H_α} = convex cone in H_α generated by $(\sigma \cap H) - \alpha$



$k[\sigma \cap \mathbb{Z}^d]$ is SFR $\Longrightarrow$

$k[\sigma_{H_\alpha} \cap \mathbb{Z}^d]$ is SFR.