Classification of noncommutative Hirzebruch surfaces

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April 11, 2024, SLMath, University of California, Berkeley, U.S.A.

Contents:

$k = \overline{k}$, $\mathrm{char} k = 0$.

X, Y : smooth projective schemes over k.

Based on joint work with Shinnosuke Okawa and Kazushi Ueda (MOU).

1 Commutative $\mathbb{P}^1$-bundles

- $\operatorname{Mod} X$: the category of quasi-coherent sheaves on X.
- $\operatorname{mod} X$: the category of coherent sheaves on X.

Definition

(1) Z is a $\mathbb{P}^1$-bundle over X $:\Longleftrightarrow \exists \mathcal{E} \in \operatorname{mod} X$ locally free of rank 2 s.t. $Z \cong \mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} S_X(\mathcal{E})$ where $S_X(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over X.

(2) Z is a ruled surface $:\Longleftrightarrow Z$ is a $\mathbb{P}^1$-bundle over a curve X.

(3) Z is a Hirzebruch surface $:\Longleftrightarrow Z$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$\mathcal{E} \in \operatorname{mod} X$ is locally free of rank 2 $:\Leftrightarrow \mathcal{E}_p \cong \mathcal{O}_{X,p} \oplus \mathcal{O}_{X,p} \ \forall p \in X$

Z is a $\mathbb{P}^1$-bundle over $X \Longrightarrow \exists f : Z \to X$ a structure morphism s.t. $f^{-1}(p) \cong \mathbb{P}^1 \ \forall p \in X$.

Example

If $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$, then

$$\mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X) = \mathrm{Proj} S_X(\mathcal{O}_X \oplus \mathcal{O}_X) = \mathrm{Proj} \mathcal{O}_X[x, y] \cong X \times \mathbb{P}^1$$

where $f : X \times \mathbb{P}^1 \to X; \ (p, q) \to p$ is a structure morphism so that $f^{-1}(p) = p \times \mathbb{P}^1 \cong \mathbb{P}^1$.

$\mathcal{L} \in \operatorname{mod} X$ is invertible $:\Longleftrightarrow - \otimes_X \mathcal{L} : \operatorname{Mod} X \to \operatorname{Mod} X$ is an autoequivalence

$\operatorname{Pic} X := \{\mathcal{L} \in \operatorname{mod} X \mid \mathcal{L}$ is invertible$\}$

Theorem

Let $\mathcal{E}, \mathcal{E}' \in \operatorname{mod} X$ be locally free of rank 2.
$\exists \mathcal{L} \in \operatorname{Pic} X$ s.t. $\mathcal{E}' \cong \mathcal{E} \otimes_X \mathcal{L} \Longleftrightarrow \mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$.

Lemma

(1) $\operatorname{Pic}\mathbb{P}^1 = \{\mathcal{O}_{\mathbb{P}^1}(a) \mid a \in \mathbb{Z}\}$.

(2) $\mathcal{E} \in \operatorname{mod}\mathbb{P}^1$ is locally free of rank 2 $\iff \exists a, b \in \mathbb{Z}$ s.t. $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$.

Corollary

Z is a Hirzebruch surface $\iff Z \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))\ \exists d \in \mathbb{N}$.

Definition

$\mathbb{F}_d := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$: Hirzebruch surface of degree d.

$\mathbb{F}_0 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

2 Noncommutative $\mathbb{P}^1$-bundles

Let R, S be commutative rings.

- $\operatorname{Mod} R$: the category of R-modules
- $\operatorname{BiMod}(R\text{-}S)$: the category of R-S bimodules

There are two ways to characterize an R-S bimodule:

(a) $\operatorname{BiMod}(R\text{-}S) \cong \operatorname{Mod}(R \otimes S) \cong \operatorname{Mod}(\operatorname{Spec}(R \otimes S)) \cong \operatorname{Mod}(\operatorname{Spec} R \times \operatorname{Spec} S)$

(b) $\operatorname{BiMod}(R\text{-}S) \cong \{- \otimes_R M : \operatorname{Mod} R \rightleftarrows \operatorname{Mod} S : \operatorname{Hom}_S(M, -) \mid$ adjoint pair of functors$\}$

Definition

(a) Let $\mathcal{E} \in \operatorname{mod}(X \times Y)$, and
$W := \operatorname{Supp} \mathcal{E} = \{p \in X \times Y \mid \mathcal{E}_p \neq 0\} \subset X \times Y$.
$\mathcal{E}$ is a sheaf X-Y bimodule if the restrictions of the projections
$u := pr_1|_W : W \to X, v := pr_2|_W : W \to Y$ are both finite.

(b) $\mathcal{E}$ is an X-Y bimodule if $\mathcal{E}$ is an adjoint pair of functors

$$- \otimes_X \mathcal{E} : \operatorname{Mod} X \rightleftarrows \operatorname{Mod} Y : \mathcal{H}om_Y(\mathcal{E}, -).$$

- $\operatorname{bimod}(X\text{-}Y)$: the category of sheaf X-Y bimodules
- $\operatorname{BiMod}(X\text{-}Y)$: the category of X-Y bimodules

Theorem [Van den Bergh (2012)]

There exists a fully faithful functor $\operatorname{bimod}(X\text{-}Y) \to \operatorname{BiMod}(X\text{-}Y)$.

Definition

(1) Z is a noncommutative $\mathbb{P}^1$-bundle over X $:\Longleftrightarrow \exists \mathcal{E} \in \mathrm{bimod}(X\text{-}X)$ locally free of rank 2 s.t. $Z \cong \mathbb{P}_X(\mathcal{E}) := \mathrm{Proj_{nc}} S_X(\mathcal{E})$ where $\mathrm{Proj_{nc}} S_X(\mathcal{E})$ is the "noncommutative projective scheme" associated to the "noncommutative symmetric algebra" $S_X(\mathcal{E})$ of $\mathcal{E}$ over X.

(2) Z is a noncommutative ruled surface $:\Longleftrightarrow$ Z is a noncommutative $\mathbb{P}^1$-bundle over a curve X.

(3) Z is a noncommutative Hirzebruch surface $:\Longleftrightarrow$ Z is a noncommutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$\mathcal{E} \in \mathrm{bimod}(X\text{-}X)$ is locally free of rank 2 $:\Longleftrightarrow$
$(\mathcal{O}_X \otimes_X \mathcal{E})_p \cong \mathcal{O}_{X,p} \oplus \mathcal{O}_{X,p}\ \forall p \in X$

Theorem

Let $\mathcal{E}, \mathcal{E}' \in \operatorname{bimod}(X\text{-}X)$ be locally free of rank 2.
$\exists \mathcal{L}_1, \mathcal{L}_2 \in \operatorname{bimod}(X\text{-}X)$ invertible s.t. $\mathcal{E}' \cong \mathcal{L}_1 \otimes_X \mathcal{E} \otimes_X \mathcal{L}_2 \Longrightarrow$
$\mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$. (Converse??)

$\mathcal{L} \in \operatorname{bimod}(X\text{-}X)$ is invertible $:\Longleftrightarrow - \otimes_X \mathcal{L} : \operatorname{Mod} X \to \operatorname{Mod} X$ is an autoequivalence

3 Classification of noncommutative Hirzebruch surfaces

Artin's conjecture (1997)

Every noncommutative integral surface Z is birationally equivalent to either

(1) a noncommutative projective plane q-$\mathbb{P}^2$ (classified),

(2) a noncommutative ruled surface $\mathbb{P}_X(\mathcal{E})$ (X is a commutative curve),

(3) a noncommutative surface which is finite over its center.

Aim

To classify noncommutative ruled surfaces $\mathbb{P}_X(\mathcal{E})$:

(1) Classify commutative curves X (classical).

(2) Classify locally free $\mathcal{E} \in \mathrm{bimod}(X\text{-}X)$ of rank 2 for each commutative curve X.

Setup

$\mathcal{E} \in \mathrm{bimod}(X\text{-}Y)$ locally free of rank 2,

$\iota : W := \mathrm{Supp}\,\mathcal{E} \to X \times Y$ embedding,

$u := pr_1|_W : W \to X, v := pr_2|_W : W \to Y$.

- $\mathrm{CM}(W) := \{\mathcal{U} \in \mathrm{mod}\, W \mid \mathcal{U}$ is maximal Cohen-Macaulay$\}$

Lemma

$\exists! \mathcal{U} \in \mathrm{CM}(W)$ s.t. $\iota_* \mathcal{U} \cong \mathcal{E}$.

Aim

Classify $(W, \mathcal{U} \in \mathrm{CM}(W))$ instead of $\mathcal{E} \in \mathrm{bimod}(X\text{-}Y)$.

Theorem [MOU]

If $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$, then one of the following cases occur:

(1) W is integral,
u, v are isomorphisms, and
$\mathcal{U}$ is locally free of rank 2.

(2) W is integral,
$\deg u = \deg v = 2$, and
$\mathcal{U}$ is isomorphic to $\mathcal{O}_W$ on an open dense subset of W.

(3) $W = W_1 \cup W_2$ is reduced with two irreducible components,
$u|_{W_i}, v|_{W_i}$ are isomorphisms for $i = 1, 2$, and
$\mathcal{U}$ is isomorphic to $\mathcal{O}_W$ on an open dense subset of W.

(4) W is irreducible and non-reduced,
$u|_{W_{red}}, v|_{W_{red}}$ are isomorphisms, and
$\mathcal{U}$ is isomorphic to $\mathcal{O}_W$ on an open dense subset of W.

From now on, we focus on the case $X = Y = \mathbb{P}^1$.

For $W, W' \subset \mathbb{P}^1 \times \mathbb{P}^1$, we define
$W \sim W' :\Longleftrightarrow \exists \tau_1, \tau_2 \in \mathrm{Aut}\mathbb{P}^1$ s.t. $(\tau_1 \times \tau_2)(W) = W'$.

Lemma

Let $\mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ be locally free of rank 2, $W = \mathrm{Supp}\,\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$.
$W' \sim W \Longrightarrow \exists \mathcal{E}' \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ locally free of rank 2 s.t.

$$\mathrm{Supp}\,\mathcal{E}' = W' \text{ and } \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}').$$

Aim

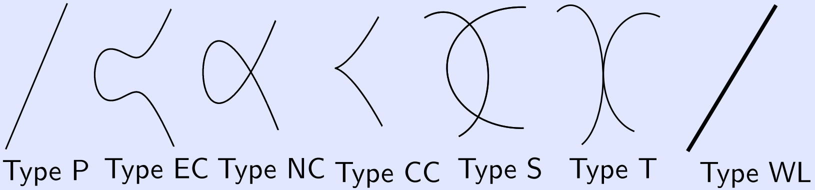
To classify noncommutative Hirzebruch surfaces $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$:

(1) Classify $W \subset \mathbb{P}^1 \times \mathbb{P}^1$ up to $\sim$.

(2) Classify $\mathrm{CM}(W)$ for each W.

Theorem [Patrick (1997), MOU]

$\forall \mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ locally free of rank 2, $W = \mathrm{Supp}\,\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a Cartier divisor of bidegree $(1,1)$ or $(2,2)$. In fact, it is equivalent to one of the following types:



Type P

Since $W \sim \Delta_{\mathbb{P}^1} := \{(p,p) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid p \in \mathbb{P}^1\}$, $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{F}_d \ \exists d \in \mathbb{N}$ (commutative Hirzebruch surface)

For the rest of the types, we define the non-invertible locus of $\mathcal{U} \in \mathrm{CM}(W)$ by

$$\mathrm{Ninv}(\mathcal{U}) = \{p \in W \mid \mathcal{U}_p \not\cong \mathcal{O}_{W,p}\} \subset \mathrm{Sing}(W).$$

We classify $\mathcal{U}$ by analyzing $\mathrm{IndCM}(\mathcal{O}_{W,p})$ (or $\mathrm{IndCM}(\widehat{\mathcal{O}_{W,p}})$) for $p \in \mathrm{Ninv}(\mathcal{U})$.

Type EC (smooth)

Since $\mathrm{Sing}(W) = \emptyset$, $\mathcal{U} \in \mathrm{Pic}\, W \cong W \times \mathbb{Z}$.

Type NC, CC, S, T (singular, reduced)

For $p \in \mathrm{Sing}(W)$, $\widehat{\mathcal{O}_{W,p}} \cong k[[x,y]]/(y^2 - x^{n+1})$ for $n = 1, 2, 3$. Using the classifications of $\mathrm{IndCM}(\widehat{\mathcal{O}_{W,p}})$, we can show that $\widehat{\mathcal{U}_p} \cong \mathrm{End}_{\widehat{\mathcal{O}_{W,p}}}(\widehat{\mathcal{U}_p})$ viewed as an $\mathrm{End}_{\widehat{\mathcal{O}_{W,p}}}(\widehat{\mathcal{U}_p})$-module.

Theorem [MOU]

If $\mathcal{U} \notin \mathrm{Pic}\, W$, then $\exists \widetilde{\mathcal{U}} \in \mathrm{Pic}\, \widetilde{W}$ s.t. $\nu_* \widetilde{\mathcal{U}} \cong \mathcal{U}$ where $\nu : \widetilde{W} := \mathcal{S}pec\, \mathcal{E}nd_W(\mathcal{U}) \to W$.

Type	$\widetilde{W}$	$\mathrm{Pic}\, \widetilde{W}$
NC, CC	$\mathbb{P}^1$	$\mathbb{Z}$
S, T	$\mathbb{P}^1 \cup \mathbb{P}^1$	$\mathbb{Z} \times \mathbb{Z}$
S, T	$\mathbb{P}^1 \sqcup \mathbb{P}^1$	$\mathbb{Z} \times \mathbb{Z}$

Type WL (non-reduced)

For $p \in \mathrm{Sing}(W) = W$, $\mathcal{O}_{W,p} \cong k[x,y]_{(x)}/(y^2)$. Using the classification of $\mathrm{IndCM}(\mathcal{O}_{W,p})$, we can show that $\mathcal{U}_p \cong (x^n, y) \lhd k[x,y]_{(x)}/(y^2)$ for some $n \in \mathbb{N}$.

Theorem [MOU]

$\sharp(\mathrm{Ninv}(\mathcal{U})) < \infty$ and $\exists \mathcal{L} \in \mathrm{Pic}\, W \cong k \times \mathbb{Z}$ s.t.

$$0 \to \mathcal{U} \to \mathcal{L} \to \mathcal{O}_{\mathrm{Ninv}(\mathcal{U})} \to 0$$

is exact.

4 Commutativity with shifts

In the commutative case, for $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b),\ a \leq b$, $\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a+k) \oplus \mathcal{O}_{\mathbb{P}^1}(b+k)$ for every $k \in \mathbb{Z}$, so $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{F}_d$ if and only if $b - a = d$.

What happens in the noncommutative case?

Definition

For $\mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ locally free of rank 2 and $k \in \mathbb{Z}$, we define $a_k, b_k \in \mathbb{Z}$ by $\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_k) \oplus \mathcal{O}_{\mathbb{P}^1}(b_k),\ a_k \leq b_k$.
We say that $\mathcal{E}$ commutes with shifts if $\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E} \cong (\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} \mathcal{E})(k)$ (ie. $a_k = a_0 + k, b_k = b_0 + k$ for every $k \in \mathbb{Z}$).

Remark

A first example of $\mathcal{E}$ which does not commute with shifts was given by Ingalls and Patrick (2002).

Let $\mathcal{U} \in \operatorname{Pic} W$, and $\mathcal{E} = \iota_* \mathcal{U}$. For $j \in \mathbb{Z}$, we define $\mathcal{U}_j := \mathcal{U} \otimes_W v^* \mathcal{O}_{\mathbb{P}^1}(j)$ and $\mathcal{E}_j := \iota_* \mathcal{U}_j$.
If $\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_k) \oplus \mathcal{O}_{\mathbb{P}^1}(b_k)$, then

$$\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E}_j \cong \mathcal{O}_{\mathbb{P}^1}(a_k + j) \oplus \mathcal{O}_{\mathbb{P}^1}(b_k + j)$$

so $\deg \mathcal{U}_j = \deg \mathcal{U} + 2j$.

Replacing $\mathcal{U}$ by $\mathcal{U}_j$ for a suitable $j \in \mathbb{Z}$, we may reduce to the cases $\deg \mathcal{U} = 0$ or $\deg \mathcal{U} = 1$ in computing $\{(a_k, b_k)\}_{k \in \mathbb{Z}}$.

Lemma

If $\mathcal{U} \in \operatorname{Pic}^0 W$ is "tame", then

$$\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} \mathcal{E} \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} & \text{if } \mathcal{U} \cong \mathcal{O}_W \\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{if } \mathcal{U} \ncong \mathcal{O}_W. \end{cases}$$

Theorem [MOU]

All possible $\{(a_k, b_k)\}_{k\in\mathbb{Z}}$ are computed in terms of $\mathcal{U}$.

Case 1

If W is irreducible (Type EC, NC, CC, WL), and $\mathcal{U} \in \operatorname{Pic} W$, then

$$(a_k, b_k) = \left(\frac{\deg \mathcal{U}}{2} + k - 2, \frac{\deg \mathcal{U}}{2} + k\right), b_k - a_k = 2$$

if $\mathcal{U} \simeq u^*\mathcal{O}_{\mathbb{P}^1}(-k) \otimes_W v^*\mathcal{O}_{\mathbb{P}^1}(j)$ $(\exists j \in \mathbb{Z})$ (so $\deg \mathcal{U} \equiv 0 \bmod 2$),

$$(a_k, b_k) = \left(\frac{\deg \mathcal{U}}{2} + k - 1, \frac{\deg \mathcal{U}}{2} + k - 1\right), b_k - a_k = 0$$

if $\mathcal{U} \not\simeq u^*\mathcal{O}_{\mathbb{P}^1}(-k) \otimes_W v^*\mathcal{O}_{\mathbb{P}^1}(j)$ $(\forall j \in \mathbb{Z})$ and $\deg \mathcal{U} \equiv 0 \bmod 2$,

$$(a_k, b_k) = \left(\frac{\deg \mathcal{U} - 1}{2} + k - 1, \frac{\deg \mathcal{U} - 1}{2} + k\right), b_k - a_k = 1$$

if $\deg \mathcal{U} \equiv 1 \bmod 2$.

Case 2

If $W = W_1 \cup W_2$ is reducible (Type S, T), and $\mathcal{U} \in \operatorname{Pic} W$ with $\deg(\mathcal{U}|_{W_1}) \leq \deg(\mathcal{U}|_{W_2})$, then

$$\{a_k, b_k\} = \{\deg(\mathcal{U}|_{W_1}) + k, \deg(\mathcal{U}|_{W_2}) + k - 2\}$$
$$b_k - a_k = |\deg(\mathcal{U}|_{W_2}) - \deg(\mathcal{U}|_{W_1}) - 2|.$$

Case 3

If W is non-reduced (Type WL) and $\mathcal{U} \notin \operatorname{Pic} W$, then

$$\{a_k, b_k\} = \left\{\frac{\chi(\mathcal{U}) - \#(\mathrm{Ninv}(\mathcal{U}))}{2} + k, \frac{\chi(\mathcal{U}) + \#(\mathrm{Ninv}(\mathcal{U}))}{2} + k - 2\right\},$$
$$b_k - a_k = |\#(\mathrm{Ninv}(\mathcal{U})) - 2|$$

Case 4

If W is reduced (Type NC, CC, S, T) and $\mathcal{U} \notin \operatorname{Pic} W$, then

Type	$\widetilde{W}$	$\widetilde{\mathcal{U}}$ $(i \le j)$	$\{a_k, b_k\}$	$b_k - a_k$
NC, CC	$\mathbb{P}^1$	$\mathcal{O}(2i)$	$\{i+k-1, i+k\}$	1
NC, CC	$\mathbb{P}^1$	$\mathcal{O}(2i+1)$	$\{i+k, i+k\}$	0
S, T	$\mathbb{P}^1 \cup \mathbb{P}^1$	$\mathcal{O}(i,j)$	$\{i+k, j+k-1\}$	$\lvert j-i-1 \rvert$
S, T	$\mathbb{P}^1 \sqcup \mathbb{P}^1$	$\mathcal{O}(i) \sqcup \mathcal{O}(j)$	$\{i+k, j+k\}$	$j-i$

where $\nu : \widetilde{W} := \mathcal{S}pec\, \mathcal{E}nd_{\mathcal{O}_W}(\mathcal{U}) \to W$ and $\widetilde{\mathcal{U}} \in \operatorname{Pic} \widetilde{W}$ such that $\nu_* \widetilde{\mathcal{U}} = \mathcal{U}$.

Corollary

(1) If W is integral (but not of Type P), then $b_k - a_k \in \{0, 1, 2\}$ for every $k \in \mathbb{Z}$.

(2) If $\mathcal{E}$ does not commute with shifts, then

- W is integral,
- $\mathcal{U} \cong u^*\mathcal{O}_{\mathbb{P}^1}(i) \otimes_W v^*\mathcal{O}_{\mathbb{P}^1}(j)$ for some $i, j \in \mathbb{Z}$, and
- $u^*\mathcal{O}_{\mathbb{P}^1}(i) \ncong v^*\mathcal{O}_{\mathbb{P}^1}(i)$ for some $i \in \mathbb{Z}$.

For a noncommutative Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, we may define a "sequence of structure morphisms" $f_i : \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \to \mathbb{P}^1$ for $i \in \mathbb{Z}$.

Theorem [MOU]

$$\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})) = \langle f_{i+1}^* \mathcal{D}^b(\operatorname{mod} \mathbb{P}^1), f_i^* \mathcal{D}^b(\operatorname{mod} \mathbb{P}^1) \rangle$$

is a semi-orthogonal decomposition for every $i \in \mathbb{Z}$.
For $i = 0$, the dual gluing functor is given by $- \otimes_X \mathcal{E}$.

We write $\mathcal{O}(i,j) := f_{-i}^*(\mathcal{O}_{\mathbb{P}^1}(j)) \in \operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ for $i, j \in \mathbb{Z}$.

Theorem [MOU]

$$\mathcal{O}(-1,-j-1), \mathcal{O}(-1,-j), \mathcal{O}(0,-1), \mathcal{O}(0,0) =: \mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}$$

is a full strong exceptional sequence for $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}))$ if and only if $a_{-1} \geq -j-1$ (eg. $(a_0, b_0) = (0, d)$, $j = 1$, and $\mathcal{E}$ commutes with shifts). In this case, $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})) \cong \mathcal{D}^b(\operatorname{mod} R)$ where

$$R = \operatorname{End}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}(\mathcal{O}(-1,-j-1) \oplus \mathcal{O}(-1,-j) \oplus \mathcal{O}(0,-1) \oplus \mathcal{O}(0,0)).$$

We need to know (a_k, b_k) for all $k \in \mathbb{Z}$ in order to construct a "full geometric helix" for $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}))$.

5 Conjecture

Definition

(a) Z is a q-$\mathbb{F}_d$ $:\Longleftrightarrow Z \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})\ \exists \mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ s.t. $\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k+d)\ \forall k \in \mathbb{Z}$.
(i.e. $(a_0, b_0) = (0, d)$ and $\mathcal{E}$ commutes with shifts).

(b) Z is a q-$(\mathbb{P}^1 \times \mathbb{P}^1)$ $:\Longleftrightarrow Z \cong \mathrm{Proj}_{\mathrm{nc}} S$ for some 3-dimensional cubic AS-regular $\mathbb{Z}$-algebra S.

(c) Z is a q-$Q \subset q$-$\mathbb{P}^3$ $:\Longleftrightarrow Z \cong \mathrm{Proj}_{\mathrm{nc}} S/(g)$ for some 4-dimensional quadratic AS-regular algebra S and an (irreducible) regular normal element $g \in S_2$.

Theorem [Van den Bergh (1996)]

A "generic" q-$Q \subset q$-$\mathbb{P}^3$ is isomorphic to some q-$\mathbb{F}_0$.

Theorem [Van den Bergh (2011)]

Every q-$(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to some q-$Q \subset q$-$\mathbb{P}^3$.

Theorem [M-Ueyama (2021), M-Nyman (preprint)]

A "standard" q-$Q \subset q$-$\mathbb{P}^3$ is isomorphic to some q-$(\mathbb{P}^1 \times \mathbb{P}^1)$.

Theorem [MOU]

A "generic" q-$\mathbb{F}_0$ is derived equivalent to some q-$(\mathbb{P}^1 \times \mathbb{P}^1)$, and vice versa.

Conjecture

There are 1-1 correspondences??

$$
\begin{array}{ccc}
 & \{q\text{-}\mathbb{F}_0\} & \\
\swarrow\!\!\!\nearrow \ (?) & & \nwarrow \\
\{q\text{-}(\mathbb{P}^1 \times \mathbb{P}^1)\} & \rightleftarrows & \{q\text{-}Q \subset q\text{-}\mathbb{P}^3\}
\end{array}
$$