40+ years of knot theory and 3-manifolds

celebrating 40 years of MSRI/SLMath

Low-dimensional topology: the study of manifolds in dimension ≤ 4.

What's special about low dimensions?

- $2+2 \geq 4, 3, 2$

  $\Rightarrow$ surfaces aren't always embedded in dimensions ≤ 4.

- $PSL_2\mathbb{C} \cong PSO(3,1;\mathbb{R})$
- $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ $\quad \binom{4}{2} = \binom{3}{2} + \binom{3}{2}$

These facts play special roles in low dims.

Knots are important for the study of 3- and 4- dimensional manifolds

I will discuss three revolutions in 3-manifold topology emphasizing knot theory where possible

- Thurston's geometrization conjecture '82
  (proved for knot complements earlier)
- The Jones polynomial '84
- Donaldson invariants of 4-manifolds & Floer invariants of 3-manifolds '88

I will say a bit about each invariant and subsequent developments.

Then I will ask what are the connections between them?

Revolution # 1 : Thurston's study of geometric structures, including foliations, and his formulation of the geometrization conjecture

Thurston identified 8 geometric structures in 3D, modeled on homogeneous Riemannian metrics.

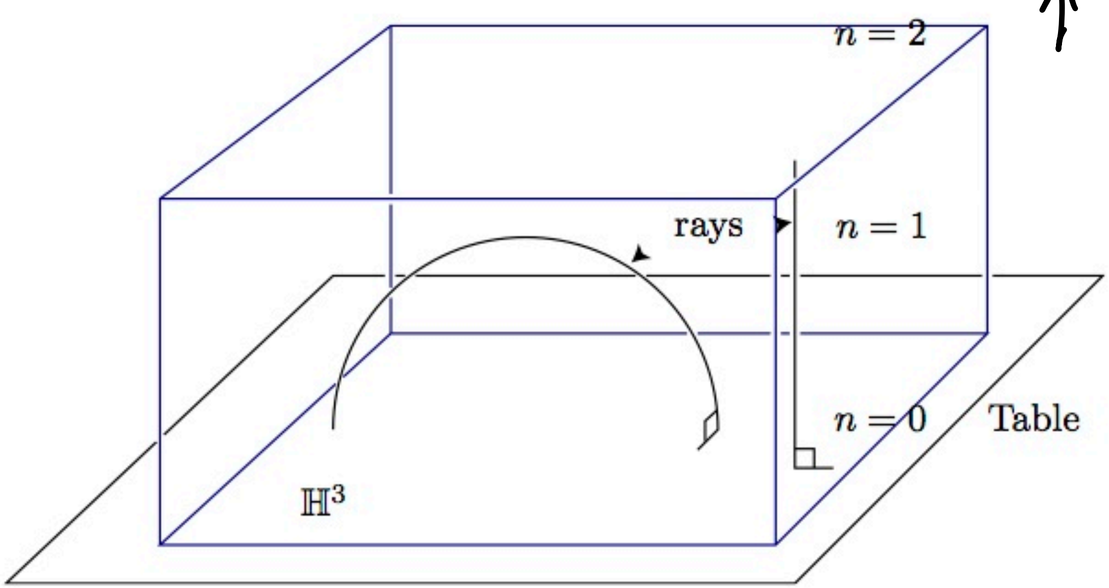
The most mysterious is hyperbolic geometry.

What is 3-dimensional hyperbolic geometry?

Consider a chunk of glass sitting on a table, so that the speed of light n is proportional to the height above the table. Then light will follow a minimal path in the glass which is a semicircle or line perpendicular to the tabletop.

point at infinity

$PSL_2\mathbb{C} \circlearrowleft$



$\hat{\mathbb{C}} = \mathbb{CP}^1$

$\frac{az+b}{cz+d}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}$$

This gives a physical approximation of the upper half space model of hyperbolic space. Hyperbolic distance is measured in the minimal time it takes for light to get from point a to point b.

Hyperbolic manifolds

The orientation-preserving isometry group of $\mathbb{H}^3$ is $PSL_2(\mathbb{C})$. The action of this group on $\mathbb{CP}^1$ extends to $\mathbb{H}^3$.

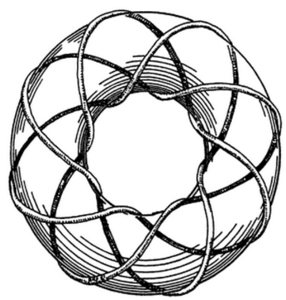
Consider a discrete subgroup $\Gamma < PSL_2(\mathbb{C})$ without any elements of finite order (discrete is equivalent to the idenity element being isolated). Then $\mathbb{H}^3/\Gamma$ is a 3-dimensional manifold admitting a complete hyperbolic metric.

$SO(3) \backslash PSL_2 \mathbb{C} / \Gamma$

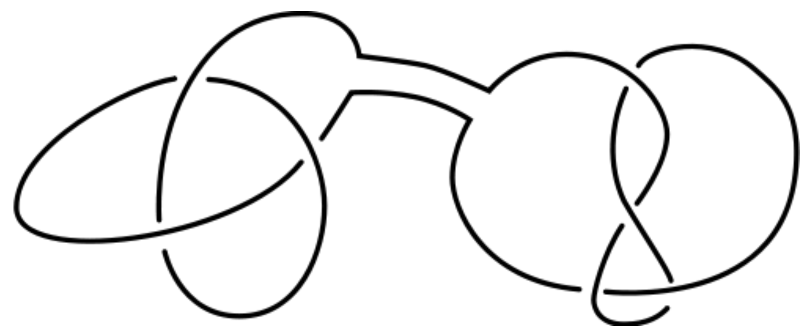
Conversely, any 3-dimensional complete Riemannian manifold with constant sectional curvature -1 will be realized in such a fashion, with Γ unique up to conjugacy in $PSL_2(\mathbb{C})$.

Knot complements

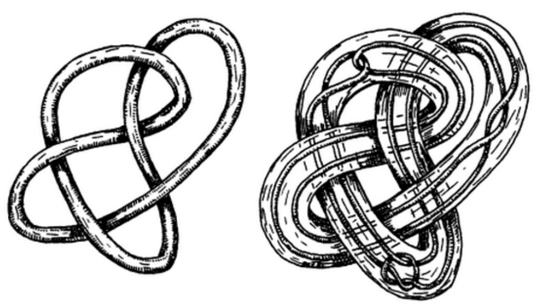
A fundamental discovery of Bill Thurston in the 1970s is that "most" knot complements admit a geometry modeled on hyperbolic geometry. Trichotomy: torus, satellite, hyperbolic



Torus knot



Connect sum (satellite)

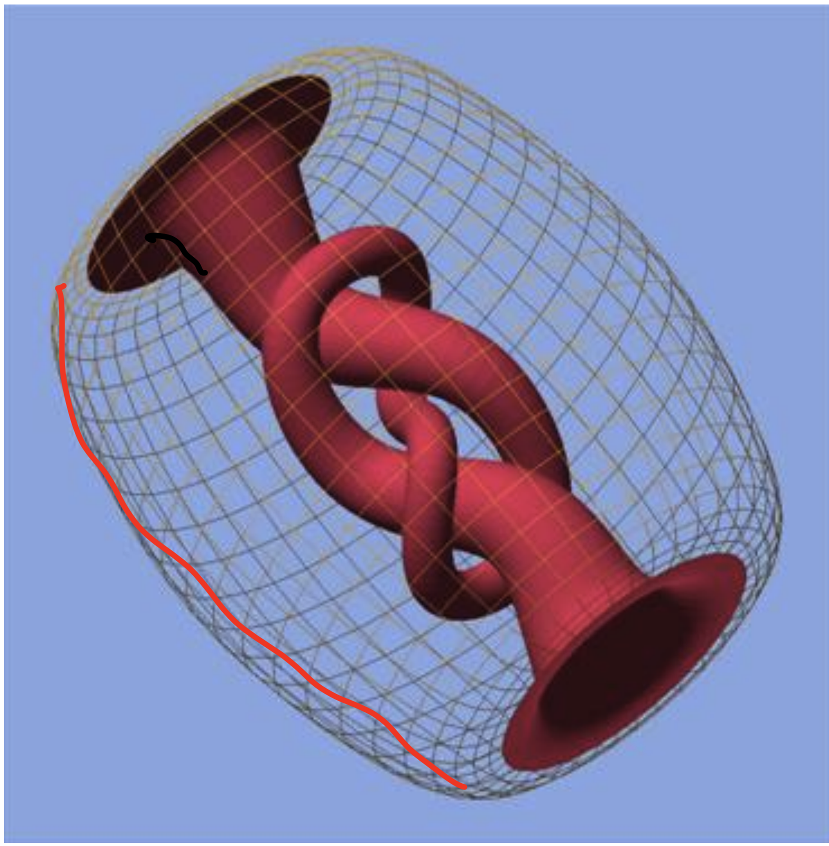


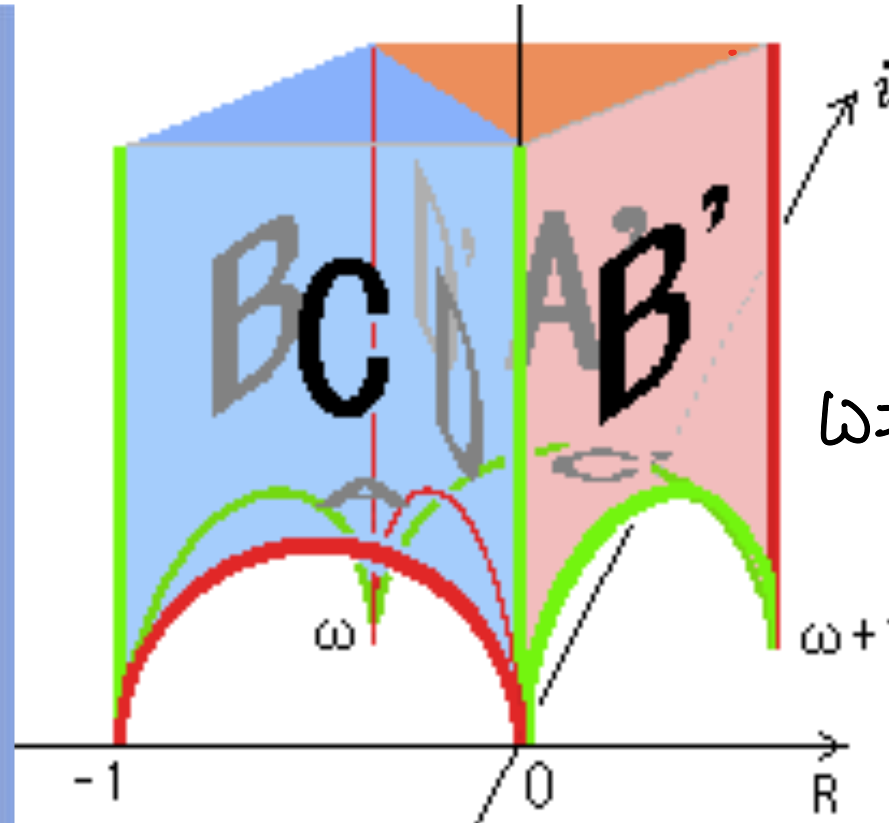
Satellite knot

Hyperbolic knot

Hyperbolic volume

A region in the upper half-space has a *hyperbolic volume*, obtained by integrating $1/n^3$ over the region.





$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

The hyperbolic volume of this knot complement $= 2.0298\ldots = \sqrt{3}(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \cdots)$.

$\zeta_{\mathbb{Q}(\sqrt{-3})}(2)$

MSRI & Thurston's legacy

1984-5 Low-dimensional topology program (organizer)

1988-9 Combinatorial group theory program

Gromov's essay on hyperbolic groups was published by MSRI and the ideas played a role in solving several of Thurston's conjectures.

1992-7 Thurston was the director of MSRI

Organized summer graduate program where he demonstrated SnapPea to me

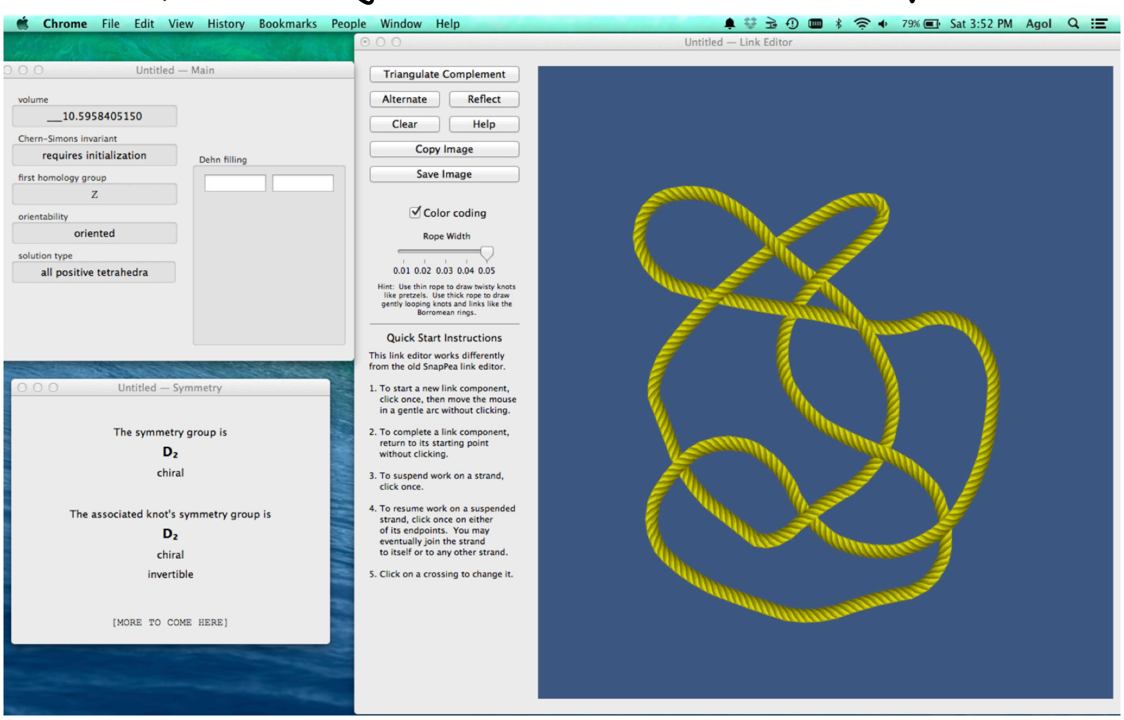
1995 complex dynamics & hyperbolic geometry program

1996-7 low-dimensional topology

2003 Conference on Ricci Flow & Perelman's proof of the geometrization theorem

2007 Teichmuller theory & Kleinian groups

2026 Topological & geometric structure in low dimensions



Revolution # 2: The Jones polynomial '84

Discovered by Jones during the course of classifying von Neumann algebras, certain infinite dimensional operator algebras.

$V(L,t)$, L an oriented link, is a Laurent polynomial in t.

skein relation.

$$t^{-1}V(\nearrow\!\!\!\!\!\nwarrow) - tV(\nwarrow\!\!\!\!\!\nearrow) = (t^{1/2} - t^{-1/2})V(\uparrow\uparrow)$$

$\Delta(L)$ Alexander polynomial satisfies

$$\Delta(\nearrow\!\!\!\!\!\nwarrow) - \Delta(\nwarrow\!\!\!\!\!\nearrow) = (t^{1/2} - t^{-1/2})\Delta(\uparrow\uparrow)$$

During the 1984-5 MSRI programs, several people noticed the similarity and simultaneously defined the HOMFLY polynomial (Ocneanu & Millet were MSRI members along with Jones)

$P_L(a,z)$

$$aP(\nearrow\!\!\!\!\!\nwarrow) - a^{-1}P(\nwarrow\!\!\!\!\!\nearrow) = z\,P(\uparrow\uparrow)$$

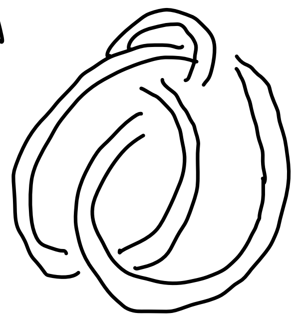
(+ a normalization for the unknot).

During the '84-5 MSRI year, Kauffman discovered the Kauffman bracket polynomial, which is a reparameterization of the Jones polynomial up to an overall power of A.

$$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$$

Colored Jones polynomials (roughly)

K^N



N-cabling of K

$V(K^N)$ is also a knot invariant.

A modification gives $V_N(K,t)$, the Nth colored Jones polynomial.

E.g.
$$J_N(E;q) = \sum_{k=0}^{N-1} \prod_{j=1}^{k} \left(q^{(N-j)/2} - q^{-(N-j)/2}\right)\left(q^{(N+j)/2} - q^{-(N+j)/2}\right)$$

E the figure 8 knot, J_N is a normalization of V_N.

MSRI /SLMath

1984 K-theory, index theory, operator algebras

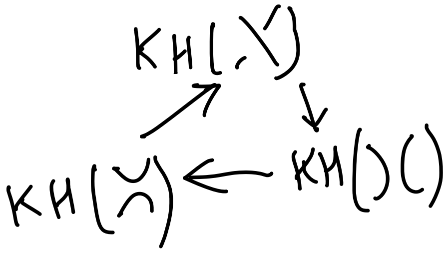
2020 Quantum symmetries (reunion 2024)

<u>Khovanov homology</u> In 1999 Mikhail Khovanov defined a chain complex $[D]$ vector spaces associated to a link diagram D so that the homology of $H_*([D])$ is an invariant of the link. $KH(L)$.

Moreover, the graded Euler characteristic

$$\sum_i (-1)^i \dim KH^{i,j}(L)\, t^j = V(L,t).$$ "categorifies"

$KH(L)$ satisfies an exact triangle



"categorifies" the Kauffman skein relation

Revolution #3: Donaldson theory & Floer homology.

In 1990 Donaldson defined invariants of 4-manifolds based on moduli spaces of self-dual instantons (gauge theory)

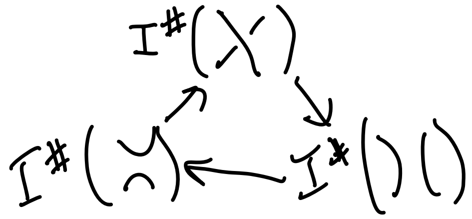
In 1988 Floer defined a graded vector space invariant of certain 3-manifolds M^3 $HF(M)$ so that if X is a 4-manifold with $\partial X = M$, then the Donaldson invariants of X take values in $HF(M)$.

Floer also introduced a (mod 4) graded vector space invariant of a knot in a homology sphere.

In 2010 Kronheimer & Mrowka introduced another grading into Floer's knot invariant and showed that the euler characteristic is the Alexander polynomial of the knot (inspired by analogous results of Ozvath-Szabo & Rasmussen).

They prove that the maximal non-zero grading is the Seifert genus of the knot and hence this detects the unknot. Analogous to a result of Ni they show that this also detects if the knot is fibered.

Exact triangle (KM 2011)



(cf. Manolescu)

MSRI / SLMath has played an important role in the development of these invariants

1996-7 Low-dimensional topology

2010 Homology theories of knots & links

2022 Floer homotopy theory

What are the connections between these three approaches to knot theory & 3-manifold topology?

Mostly conjectural

Donaldson

Kronheimer–Mrowka "Khovanov homology is an unknot detector"

L-space conjecture; Knot Floer homology detects some knots

Thurston $\xleftrightarrow[\text{(Kashaev '97)}]{\text{Volume conjecture}}$ Jones

Ideas of Thurston and his school play an important role in Floer invts., e.g. sutured manifolds (Gabai), open book decompositions, confoliations (Eliashberg-Thurston)

Thm. (Friedl-Vidussi, Taubes conjecture)
$S^1 \times M^3$ is symplectic $\iff$ M fibers over S^1.

$\Leftarrow$ Thurston

$\Rightarrow$ uses techniques developed to prove Thurston's virtual fibering conjecture & gauge theory (Gr = SW & relation with Alexander poly. Taubes).

The volume conjecture (reformulated by Murakami-Murakami, '01)

Let K be a hyperbolic knot, $\mathrm{Vol}(S^3 - K)$ the hyperbolic volume

$$\lim_{N \to} \frac{2\pi}{N} \log \left| \tilde{V}_N(K, e^{\pi i/N}) \right| = \mathrm{Vol}(S^3 - K)$$

$$\tilde{V}_N(K, t) = \frac{V_N(K, t)}{V_N(O, t)}$$

If true, it would demonstrate the power of the V_N invariants, they would distinguish hyperbolic knots up to finite ambiguity.

Would give a very difficult to compute formula for the hyperbolic volume!

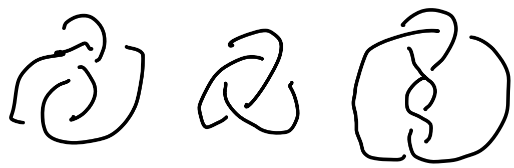
Relates the Jones & Thurston schools of knot theory.

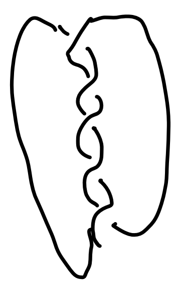
Many generalizations & related conjectures.

Khovanov homology is an unknot detector (Kronheimer-Mrowka)

They proved that there is a spectral sequence from Khovanov homology to knot Floer homology.
This demonstrated that Khovanov homology detects the unknot since knot Floer homology does.

More recently Baldwin-Sivek and others have shown that several other knots such as the figure 8, trefoil & 5_2 knots are detected by Khovanov homology





These techniques can only be pushed so far, since there are known to be knots with isomorphic Khovanov homology.

These results give connections between Donaldson/Floer & Jones schools of 3-mfd. topology.
Problem: Find a gauge-theoretic interpretation of the Jones poly. & Khovanov homology (Witten has proposals)

Knot Floer homology & surface dynamics

Ozsváth-Szabo and Rasmussen defined a version of Knot Floer homology based on Heegaard diagrams in 2003. Conjecturally isomorphic to Floer's knot invariant,

Osvath-Szabo showed it detects the Seifert genus.

Ghiggini proved that it detects the trefoil & figure 8 knots.

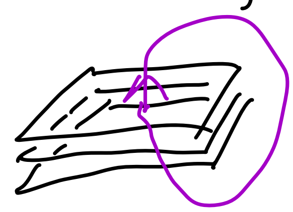
Ni proved that it detects fibering

Farber-Reynoso-Wang '22: detects cinquefoil. Their proof made use of ideas of Baldwin-Sivek, based on the thesis of Cotton-Clay, that related the # fixed points of the monodromy to the rank of knot Floer homology in the second highest grading. They used Thurston's theory of pseudoAnosov maps & train tracks to prove that certain fibered hyperbolic knots have more fixed points than the cinquefoil, & hence have different knot Floer homology.

This gives a connection between the Thurston & Donaldson schools of 3-manifold topology.

L-space conjecture Ozsváth-Szabo defined many flavors of Floer homology of closed manifolds based on Heegaard splittings. The simplest version is $\widehat{HF}(M)$, and they proved that if $|H_1(M)|<\infty$, then rank $\widehat{HF}(M) \geq |H_1(M)|$. If equality holds, then M is called an **L-space**. (e.g. lens spaces)

They proved that if M admits a taut foliation, then M is not an L-space.



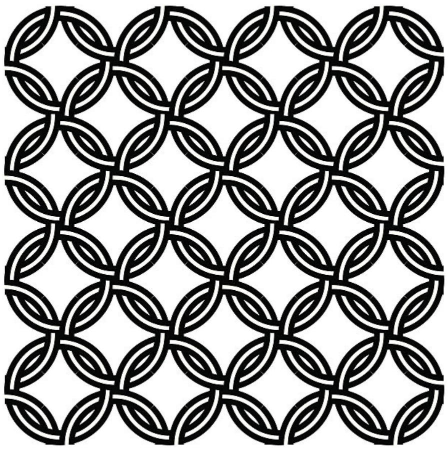
This gives another connection between the Thurston and Donaldson schools.

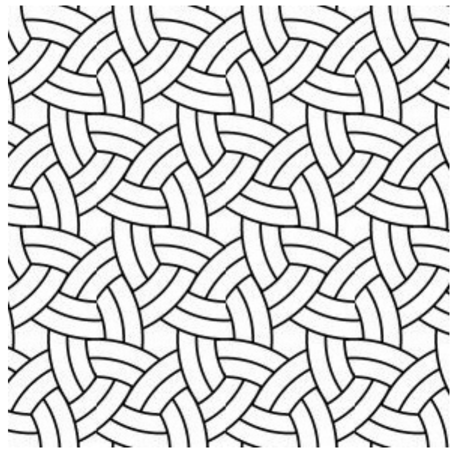
Stipsicz conjectures that for M irreducible, the converse is true: if M is not an L-space, then M admits a taut foliation.

This would be interesting if true, since it is difficult to tell when a 3-manifold admits a taut foliation (A.-Li proved $\exists$ an algorithm, but not practical).

Proved for graph manifolds in 2020 by Hanselman-Rasmussen-Rasmussen-Watson.

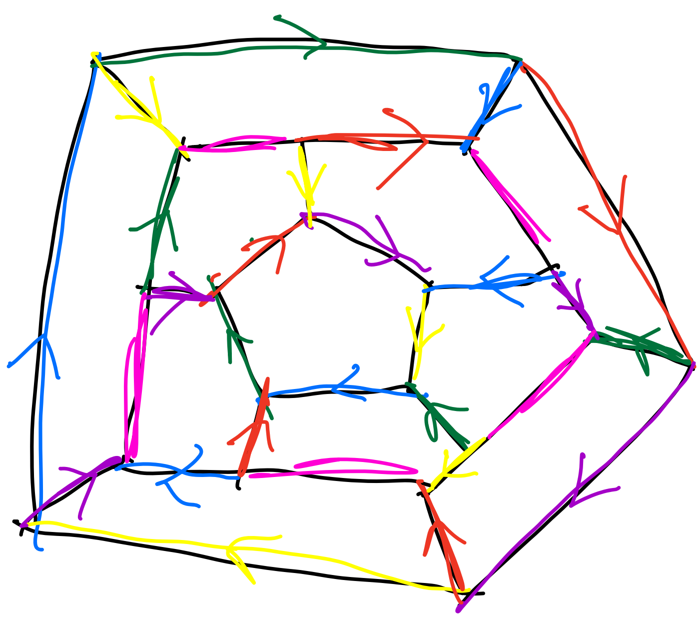
Thm. Alternating chainmail links are L-space links





Uses exact triangle for Dehn filling on these links

Lin - Lipnowski give a criterion for a hyperbolic manifold to be an L-space. They use their criterion to prove that the Seifert-Weber dodecahedral space is not an L-space. Was inaccessible by nongeometric techniques.



Conclusion: There are exciting hints and some theorems forging connections between these different approaches to 3-manifold topology. This will lead to information flowing between these subjects.

Happy birthday SLMath!

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Curved Spaces geometrygames.org

or App Store