is also an invariant of V. It is a totally real number field, of degree at most g over $\mathbb{Q}$, satisfying

$$K = \mathbb{Q}(\operatorname{tr} A) \tag{2.9}$$

for any hyperbolic element $A \in \mathrm{SL}(X, \omega)$. Moreover $K = \mathbb{Q}$ if and only if (X, ω) is the pullback of a form of genus one.

3. All generators of V lie in the same *stratum* $\Omega\mathcal{M}_g(p)$, so this too is an invariant of V.

The trace field and stratum are known for all the Teichmüller curves V we will discuss below. The lattice $\mathrm{SL}(X, \omega)$, on the other hand, is often inaccessible. Nevertheless, *topological* invariants of V, such as its Euler characteristic, can frequently be determined.

3. Billiards

We now turn to the remarkable connection between Teichmüller curves and billiards in polygons.

The first nontrivial Teichmüller curves $V \subset \mathcal{M}_g$ were discovered in 1989 by Veech. They play a key role in his proof of:

Theorem 3.1. *Billiards in a regular polygon P has optimal dynamics.*

Here *optimal dynamics* means that any unit speed billiard trajectory $\tau : \mathbb{R} \to P$ satisfies the *Veech dichotomy*; it is either

(i) *periodic*: meaning $\tau(t) = \tau(t + T)$ for some $T > 0$; or
(ii) *uniformly distributed*: meaning $\tau(\mathbb{R})$ is dense, and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau(t))\, dt = \frac{1}{\operatorname{area}(P)} \int_P f(z)\, |dz|^2$$

for any continuous function $f : P \to \mathbb{R}$.

Which alternative holds—(i) or (ii) above—depends only on the *initial slope* of the trajectory. See Figure 3.1 for examples.

In this section we describe the series of Teichmüller curves associated to regular polygons, and present the proof of Theorem 3.1, following [V1] and [Mas2]. We also summarize, in Theorem 3.9, the known examples of triangles with optimal billiards.

A striking feature of Theorem 3.1 is that it describes the behavior of *every* trajectory in P, and shows that only two, radically different types of behavior are

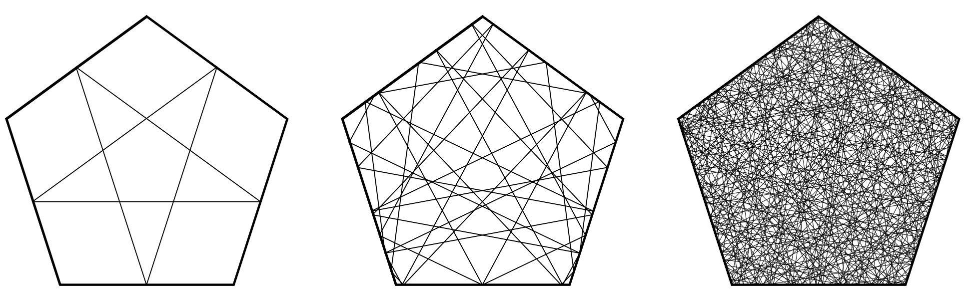

Figure 3.1. Three billiard trajectories in a regular pentagon.