The Languages of Product-Mix Auctions

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Oxford University

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Also material from Baldwin, Bichler, Fichtl, Klemperer (2022)
and Baldwin, Goldberg, Klemperer and Lock (2023)

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Bank of England Problem

After Northern Rock bank run, Bank of England urgently wants to loan funds to banks, etc., – willing to take weaker-than-usual collateral, but only in return for higher interest rate.

i.e., wanted to sell related goods to banks (loans against different kinds of collateral: "strong" (UK / US sovereign debt), "weak" (mortgage-backed securities?!), etc.

General Problem

Supplier wants to sell multiple versions of a product: multiple goods.

Goods might be divisible or indivisible. Focus on indivisible for today.

Seller costs depend on bundle of goods sold. So their preferred bundle to sell depends on prices on **all** goods.

Bidders' demand depends on prices on **all** goods.

Reason to prefer a sealed bid mechanism.

Discrete Convex Analysis approaches, and related work

Kelso and Crawford (1982), Murota and co-authors (long literature); Milgrom (2000), Ausubel (2006); Paes Leme and Wong (2015)

- Focus on **finding Walrasian equilibrium**
- Preference data either gathered dynamically or assumed already known and aggregated

Existing Approaches

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"Bidding language" approaches

Milgrom (2009); Nisan (2006); Klemperer (2008, 2010)

- Focus on **gathering bid data**
- Does set of preferences communicated in the language align with nice economic properties?

Mostly in context of "strong substitute" preferences (see later)

Our Contribution: Bidding Languages via Geometry

Bidding languages so we can build out of **simple pieces** any valuation from a given class of valuations:

Strong substitutes

All substitutes

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Auction Setting:

- (Approximately) competitive bidding behaviour
- Seller maximises efficiency.
- Seller preferences can be as rich as buyers (or richer!)

Simple Bids (Bank of England auction, Klemperer 2010)

A bid $\boldsymbol{b} = (\mathbf{r}, 1)$ represents utility $v_{\mathbf{b}}(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, valuation $v_{\mathbf{b}} : \Delta_n \to \mathbb{R}$,

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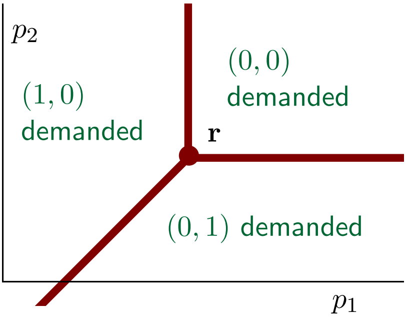

- Demand set $D_{\mathbf{b}}(\mathbf{p}) = \text{argmax}_{\mathbf{x} \in \Delta_n}(v_{\mathbf{b}}(\mathbf{p}) - \mathbf{p} \cdot \mathbf{x})$
- So demand goods maximising $r_i - p_i \geq 0$, or nothing.
- Bid for at most one unit, of good with with best price p_i relative to r_i.

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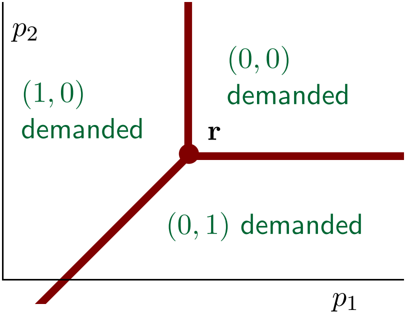

- Gul and Stacchetti (1999) "unit demand"
- Simple case of Milgrom (2009) integer assignment messages.

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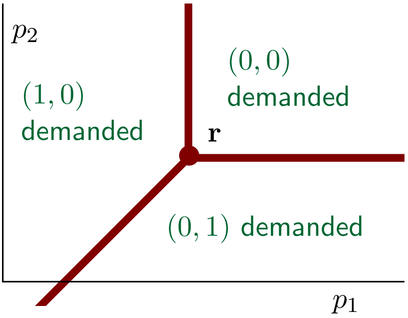

Easy to aggregate many bids.

Bid $\boldsymbol{b} = (\boldsymbol{r}, m)$ with multiplicity $m \in \mathbb{Z}_+$ aggregates m identical bids.

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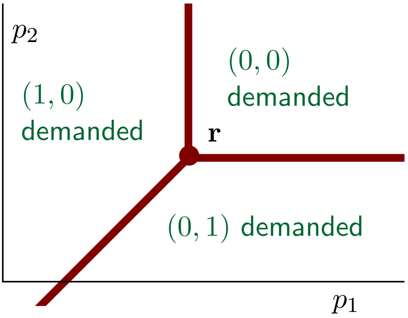

Finding market clearing price:

- Optimise individual bids via linear / integer programming
- Aggregate these linear programs by adding them up

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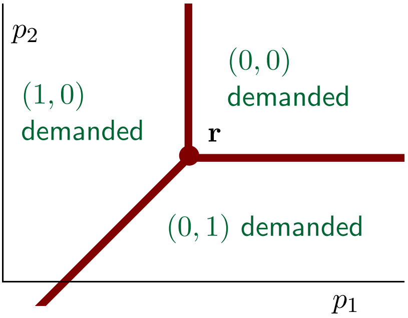

Associate "Locus of Indifference Prices" (LIP) $\mathcal{L}_{\boldsymbol{b}}$, with "facets":

- Where bidder indifferent between nothing and unit of good i
- Where bidder indifferent between good i and good j

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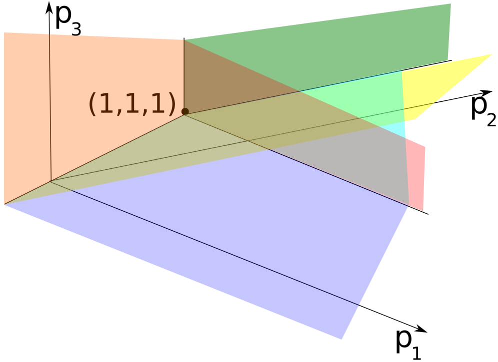

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Facets and LIPs

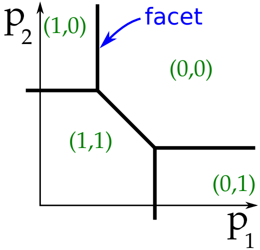
More generally, given

- Valuations $v : A \to \mathbb{R}$ on finite domain $A \subsetneq \mathbb{Z}$
- quasilinear utility $v(\mathbf{X}) - \mathbf{p} \cdot \mathbf{x}$

Identify **what is demanded where**: consider **where demand changes**.

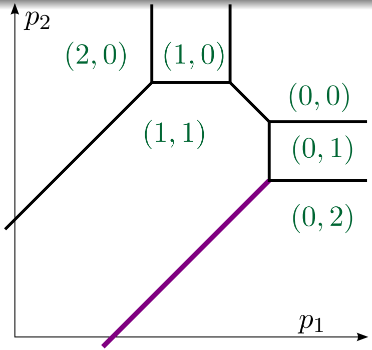
Definition: "Locus of Indifference Prices (LIP)"

$$\mathcal{L}_v = \{ \text{ prices } \mathbf{p} \in \mathbb{R}^n \text{ where } |D_v(\mathbf{p})| > 1\}.$$



Composed of linear pieces: **facets**.
Here v is indifferent between bundles uniquely demanded on either side.

What do facets of LIPs "mean"?



If $\mathbf{p}$ is in a facet then the agent is indifferent between two bundles:

$$u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = u(\mathbf{y}) - \mathbf{p}.\mathbf{y}$$

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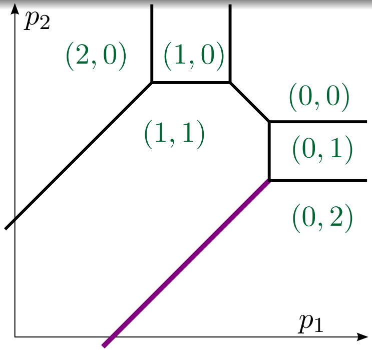

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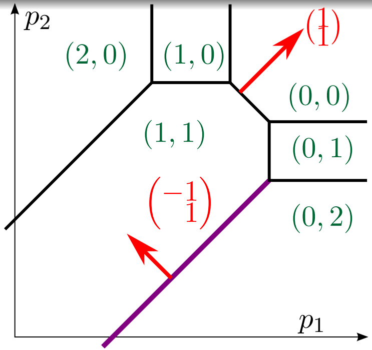

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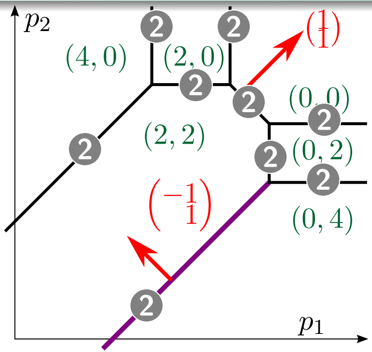

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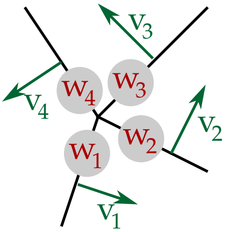
The change in bundle is in the direction normal to the facet.

The change in bundle is ‘weight $w > 0$’ times the minimal facet normal.
Work with weighted LIPs $(\mathcal{L}_u, \mathbf{w}_u)$. Facet F has weight $\mathbf{w}_u(F) \in \mathbb{Z}_{>0}$.

Economics from Geometry



Every LIP is **balanced**: around each $(n-2)$-cell, $\sum_i w_i v_i = 0$.

Theorem (Mikhalkin 2004; the Valuation-Complex Equivalence Theorem)

A weighted rational polyhedral complex of pure dimension $(n-1)$ is the LIP of a valuation **iff** *it is* **balanced**.

A LIP corresponds to an **essentially** *unique concave valuation.*

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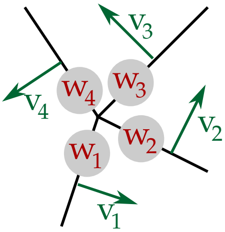

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We can depict all valuations (of a certain class)

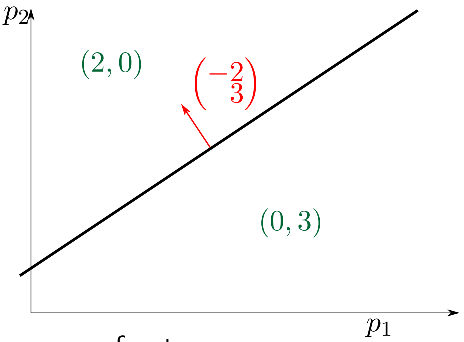
$\Leftrightarrow$

We can draw all pictures (with certain properties).

Demand types

Suppose every facet normal $\mathbf{v}$ to the LIP $\mathcal{L}_u$...

has at most one +ve, one -ve coordinate entry.



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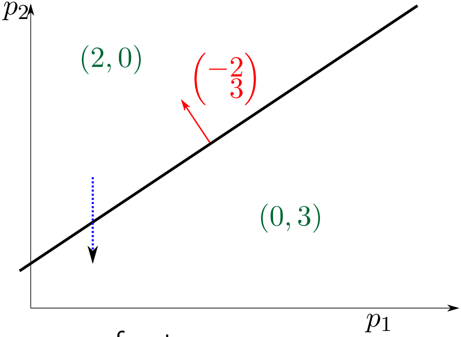

Decrease price i to cross a facet.

- Demand changes from $\mathbf{x}$ to $\mathbf{x} + \mathbf{v}$, where $\mathbf{v}$ is a facet normal.
- By the law of demand, $v_i > 0$.

$\Rightarrow$ $v_j \leq 0$ for all $j \neq i$.

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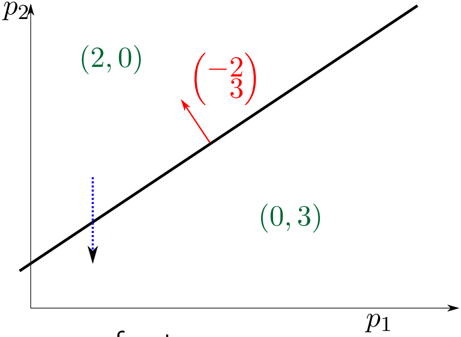

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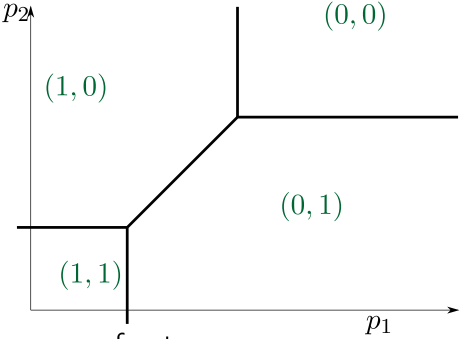

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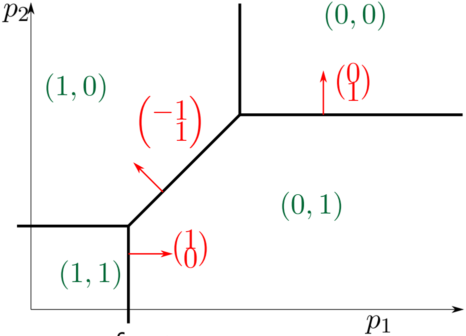

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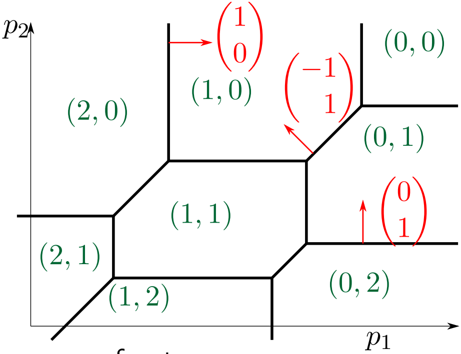

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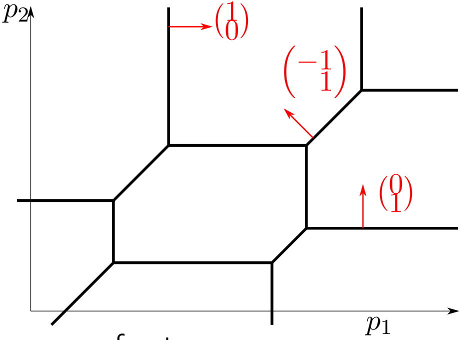

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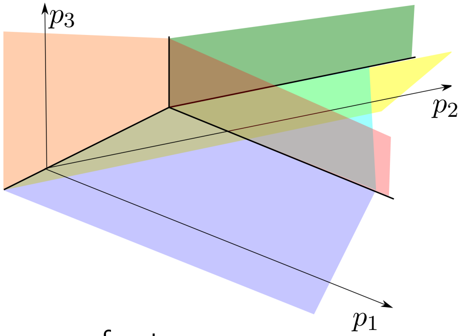

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Definition: "Demand Type"

u is of demand type $\mathcal{D}$ if every facet of $\mathcal{L}_u$ has normal in $\mathcal{D}$.

The demand type is the set of all such valuations.

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Multiple Bids Forming LIPs

A collection of positive bids $\mathbf{b} = (\mathbf{r}; m) \in \mathcal{B}$

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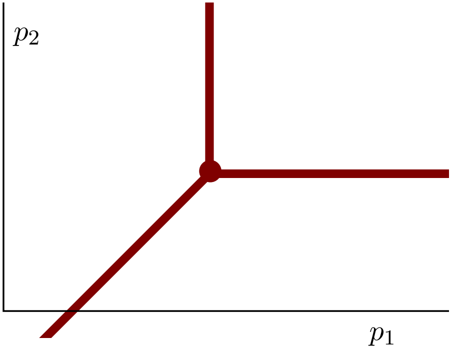

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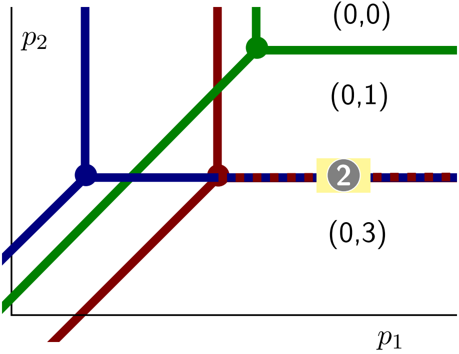

Need for the Strong Substitute Bidding Language

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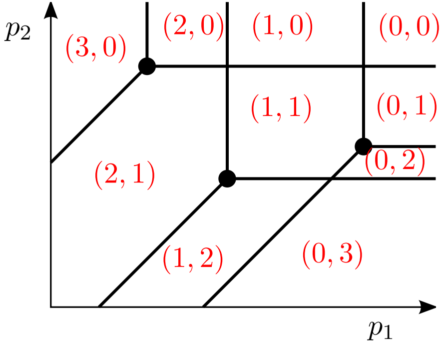

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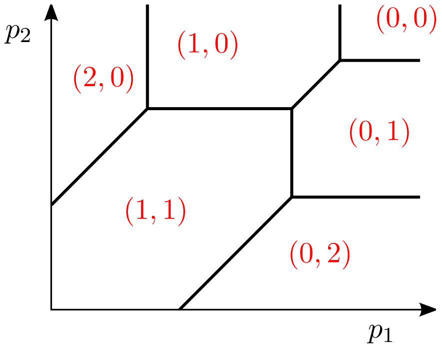

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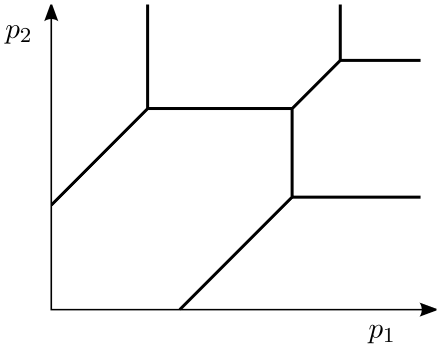

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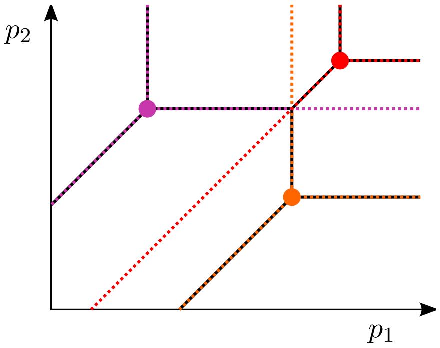

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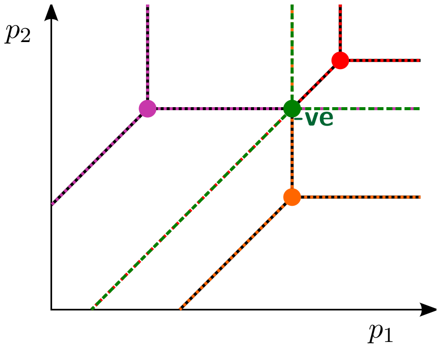

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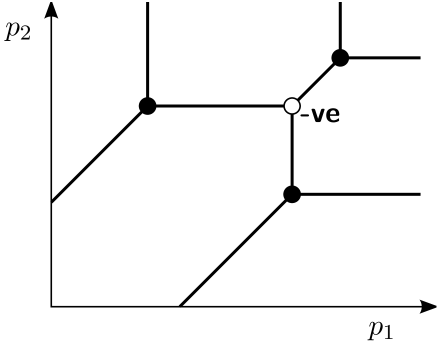
Works if we "subtract a bid"
Include bids with negative multiplicity



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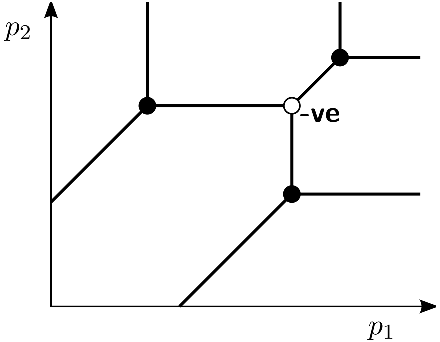
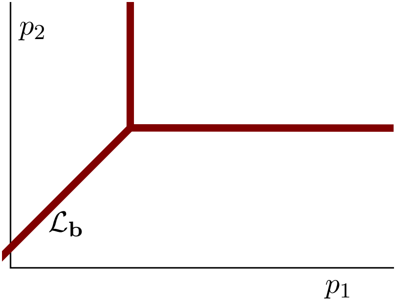


Increase the richness of the language, while keeping it relatively easy to understand, aggregate and optimise.

$\mathcal{B}$ set of bids $\mathbf{b} = (\mathbf{r}, m)$ where $m \in \mathbb{Z}$.



Aggregating $\mathbb{Z}$-weighted bids

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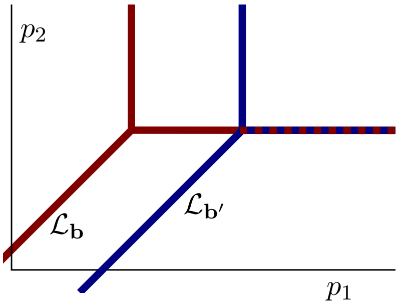

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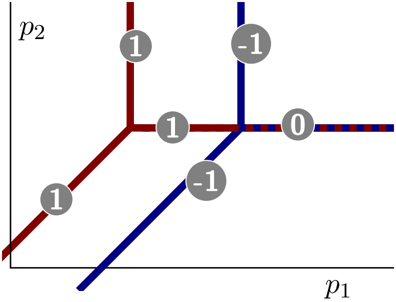
- Take the union of the LIPs.
- Add multiplicities to get facet weights.



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$\mathcal{B}$ set of bids $\mathbf{b} = (\mathbf{r}, m)$ where $m \in \mathbb{Z}$.

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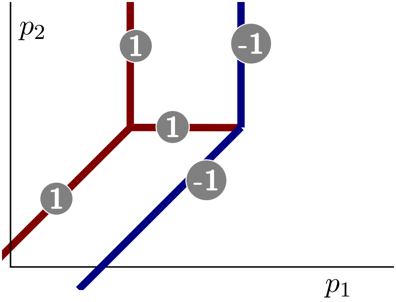

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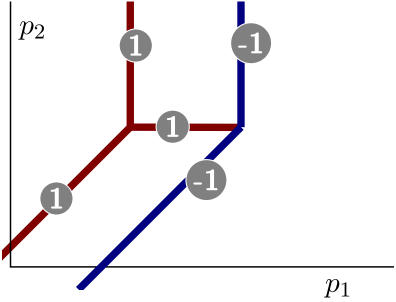

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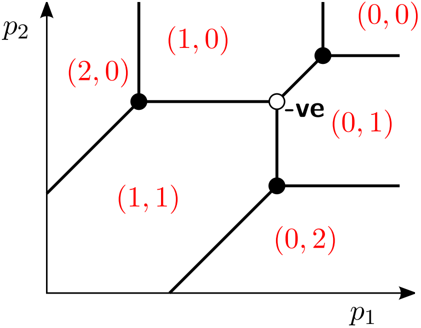

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Definition

A collection $\mathcal{B}$ of $\mathbb{Z}$-weighted bids is **valid** if $\mathbf{w}_\mathcal{B} \geq 0$.

$\mathcal{L}_{\boldsymbol{b}}$ is of strong subs demand type, so by valuation-complex equivalence theorem:

If bids are valid, they represent a strong substitute valuation.

So can define $(\mathcal{L}_\mathcal{B}, \mathbf{w}_\mathcal{B})$ from a valid set $\mathcal{B}$ of $\mathbb{Z}$-weighted bids.

Demand from Positive and Negative Bids

Translating $\mathcal{B}$ to $D_{\mathcal{B}}(\mathbf{p}) := D_{v_{\mathcal{B}}}(\mathbf{p})$ is easy when demand is unique.

$$D_{\mathcal{B}}(\mathbf{p}) = D_{\mathcal{B}^+}(\mathbf{p}) - D_{|\mathcal{B}^-|}(\mathbf{p})$$

where $\mathcal{B}^+, \mathcal{B}^-$ are positive- and negative-weighted bids in $\mathcal{B}$.

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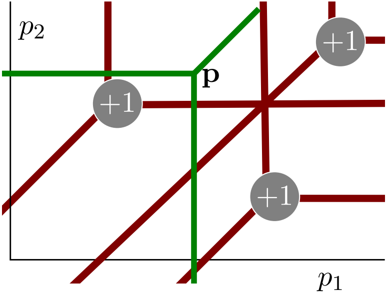

Demand goods maximising $r_i - p_i \geq 0$, or nothing.

$D_{u^{\mathcal{B}}}(\mathbf{p}) = (0, 2)$

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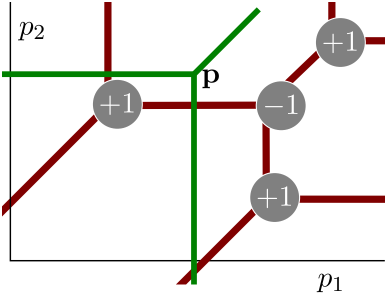

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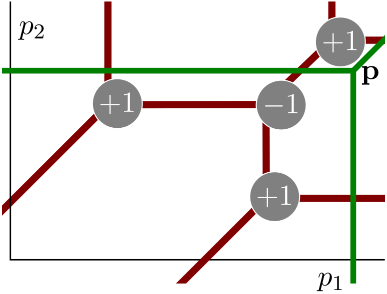

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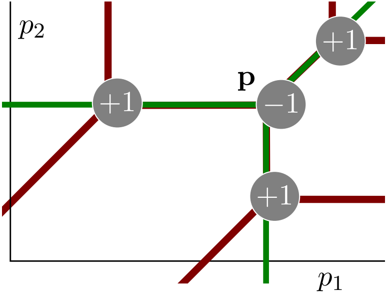


Demand goods maximising $r_i - p_i \geq 0$, or nothing.

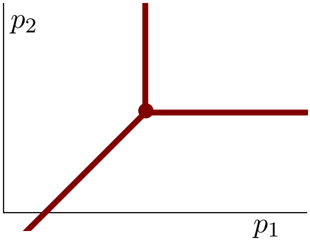
$D_{u^{\mathcal{B}}}(\mathbf{p})$
$= \{(1,0), (0,1), (1,1)\}$

In general, find all nearby unique demands and take discrete convex hull.
Use this principle to implement the auction.

Details

Extension: Bids that "shoot off the page"

If single positive bid looks like this,



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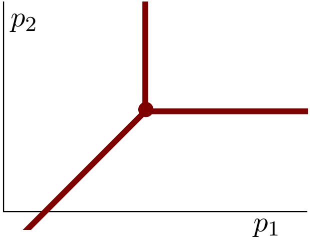
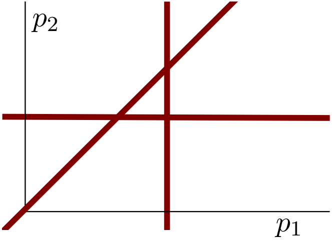
If single positive bid looks like this, how do we depict preferences like this?





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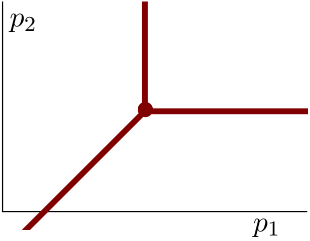

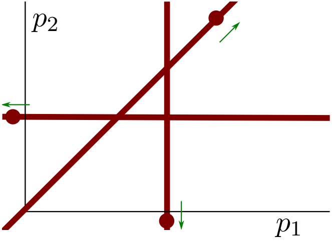

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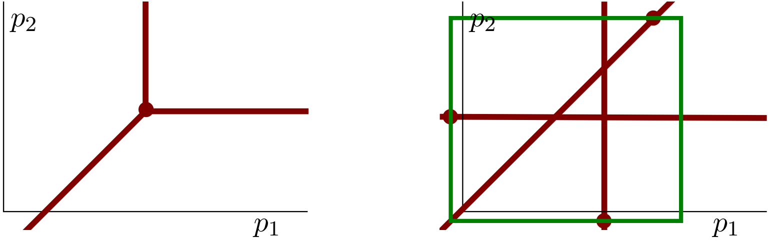

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Can approximate these with regular bids on the boundary of a "bounding box" containing all the action.

Representation of Strong Substitute Valuations

Theorem (Characterisation of Strong Substitutes)

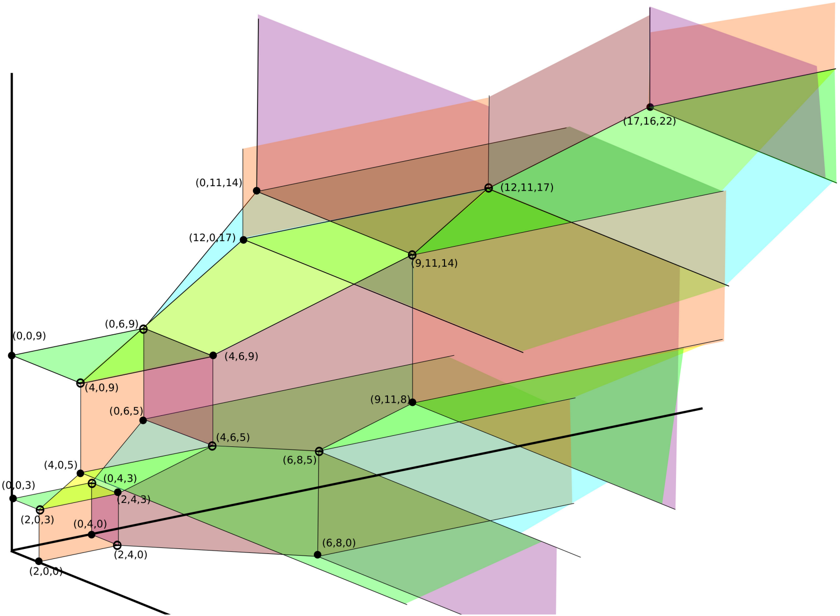
If a finite collection of positive and negative extended bids is valid then it represents a strong substitutes valuation.

If valuation $u : A \to \mathbb{R}$ is strong substitutes then it can be presented using a valid finite collection of positive and negative extended bids.

First part already seen. Sketch proof of second part follows

See also Lin and Tran (2017).

3-dimensional example



Tool 1: Valuation-complex equivalence theorem

Will use bids to draw a picture that matches $\mathcal{L}_v$.

Key tools for Proving the Representation Theorem

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Tool 2: HIPs

Extend all facets in the LIP $\mathcal{L}_v$ to create a "HIP" $\mathcal{H}_v$, a union of hyperplanes: "hyperplanes of indifference prices".

- Facet weights are inherited from $\mathcal{L}_v$, and so can be zero.
- Every hyperplane in $\mathcal{H}_v$ contains a facet of non-zero weight.

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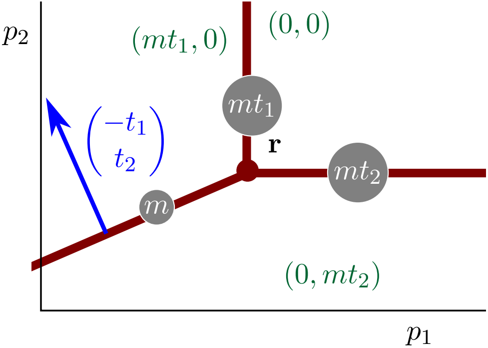
Tool 3: Balancing
The LIPs and HIPs of valuations and $\boldsymbol{p}$ bids are all balanced.
Will imply we mainly need look at one orientation of hyperplanes (except for "extended" bids).
Get the facet weights right there, and everything else follows.

More detail

All Substitutes Bidding Language

Now relax the “strong” assumption on substitutes. Use positive and negative dots **with non-trivial trade-offs** to depict all preferences for integer substitutes valuations.



Bids now have “trade-offs”: $\mathbf{b} = (\mathbf{r}; \mathbf{t}; m)$

Expressivity of the All Substitutes Language

Theorem

If a finite collection of weighted positive and negative bids is valid then it represents a substitutes valuation.

If $A \subsetneq \mathbb{Z}^n$ and $u : A \to \mathbb{R}$ is a substitutes valuation then it can be presented using a valid finite collection of weighted positive and negative bids.

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However, now competitive equilibrium is not guaranteed with integer goods. Need divisibilities or wastage.

Summary

- Need for sealed-bid auctions simultaneously selling multiple goods
- We can approach auction design using “bidding languages”
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More details of representation theorem proof

Implementation of Strong Substitutes Auction

Failure of Competitive Equilibrium for general Substitutes

L. Ausubel. An efficient dynamic auction for heterogeneous commodities. *The American economic review*, 96(3):602–629, 2006.

E. Baldwin and P. Klemperer. Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities. *Econometrica*, 87:867–932, 2019.

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Implementation of the Auction

Wish to sell bundle $\mathbf{y}$. Two phases

1. Find an equilibrium price $\mathbf{p}$ so that $\mathbf{y} \in D_{\mathcal{B}}(\mathbf{p})$.
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Address 1 & 2: "Solving Strong-Substitutes Product-Mix Auctions".
(EB, Paul Goldberg, Paul Klemperer and Edwin Lock)
Mathematics of Operations Research, 2023.

Address 1: "Strong Substitutes: Structural Properties, and a New Algorithm for Competitive Equilibrium Prices"
(EB, Martin Bichler, Maximilian Fichtl and Paul Klemperer)
Mathematical Programming, 2022.

Note that guarantee of competitive equilibrium for strong substitutes $\Rightarrow$ no worries about indivisibilities on phase 1.

Finding an Equilibrium Price Method 1 (BGKL)

- Using bids, easy to calculate aggregate indirect utility $\pi_{\mathcal{B}}(\mathbf{p}) = \max_{\mathbf{x}\in A}\{v_{\mathcal{B}}(\mathbf{x}) - \mathbf{p}\cdot\mathbf{x}\}$

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- Find "long step" of **how much** to decrease these prices, using bids.

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- Use difference of convex functions programming to find this minimum of difference of linear programmes.

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- Iteratively eliminate leaves (slightly more to this)
- Break cycles labelled by more than one bidder by ‘tweaking’ bids up slightly (requires defined order of priority). An allocation after a sufficiently small tweak is a valid un-tweaked allocation.

Demand in Strong Substitutes Bidding Language Summary

Allowing More General Substitute Trade-offs?

Suppose trade-offs are not all 1-1.

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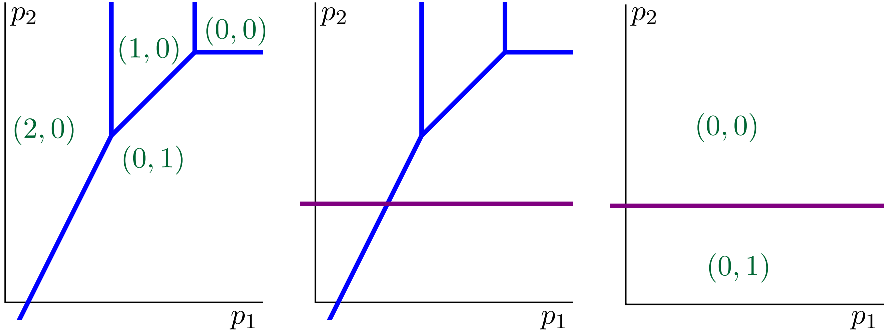

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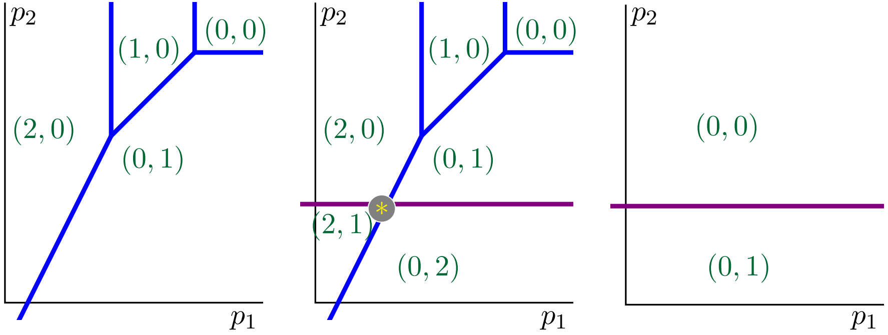
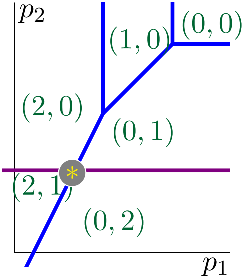
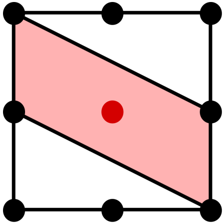
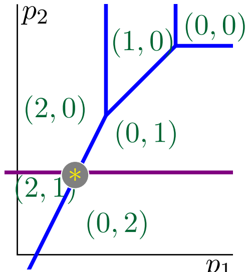
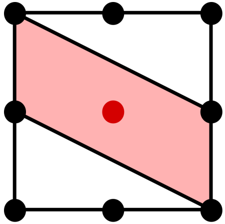


Equilibrium not guaranteed with indivisible goods:
Bundle $(1, 1)$ "should" be demanded at price *.
Weaken again to divisible goods (or allow a small number of units to be "wasted").


p_2
(1, 0)
(0, 0)
(2, 0)
(0, 1)
(2, 1)
(0, 2)
p_1

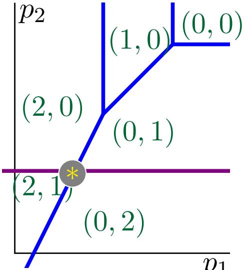


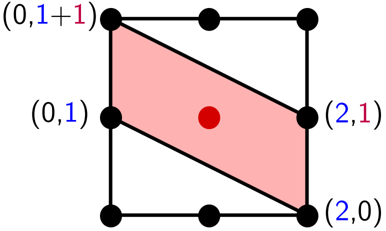




Is $(1,1)$ demanded at the star?

- At price ✻
  - purple demands $(0,0)$ or $(0,1)$
  - blue demands $(2,0)$ or $(0,1)$
- Aggregate demand set is Minkowski sum of demands
- there is no way to demand the bundle in the middle





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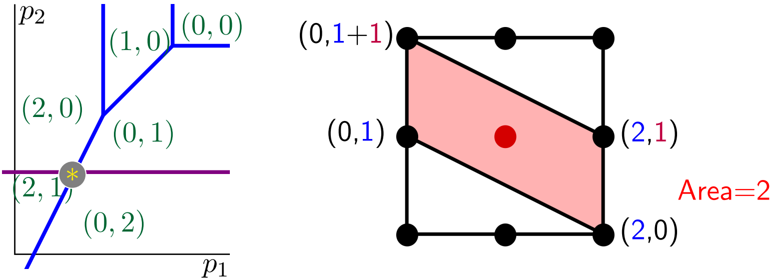
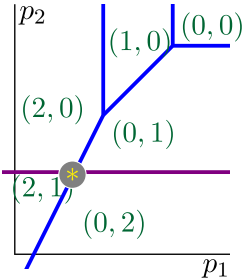
No!

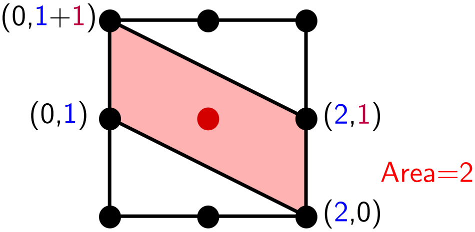
And so it is not demanded anywhere (think supporting hyperplane)

Equilibrium and Unimodularity: (BK19)



- There *exists* a non-vertex bundle because the shape's *area* is > 1
- The area is the abs value of the determinant of vectors along its edges
- Avoid problems iff all sets of n demand type vectors have det ± 1 or 0.





- There *exists* a non-vertex bundle because the shape's *area* is > 1
- The area is the abs value of the determinant of vectors along its edges
- Avoid problems iff all sets of n demand type vectors have det ± 1 or 0.

Equilibrium exists for all collections of valuations of a demand type, iff the set of vectors defining that type is **unimodular**.

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Given a valuation v for strong substitutes

1. Define "Bounding box" containing all vertices of $\mathcal{H}_v$.
   Write $[\mathcal{H}_v]$ to also include edges of this box.

Proof Structure for Representation Theorem

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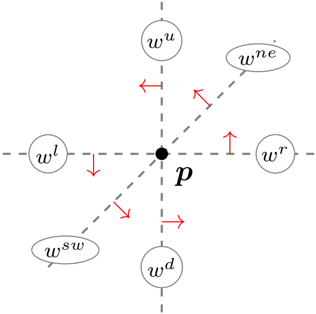
6. $D_v = D_{\mathcal{B}}$ by "facet normal times weight equals change in demand".

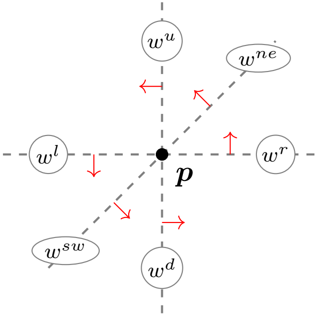
More Detail for $n = 2$: Creating the Bid List

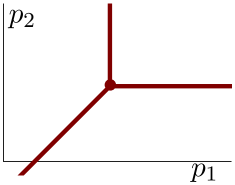
Label weights of facets adjacent to a price $\mathbf{p} \in \mathbb{R}^2$ as follows



Subscript will denote the valuation / bid(s) whose weight we are taking

More Detail: Creating the Bid List





For each vertex $\boldsymbol{p}$ of $[\mathcal{H}_v]$ that is

- inside the box, add a bid:
  - root $\boldsymbol{r} = (0, p_1, p_2)$
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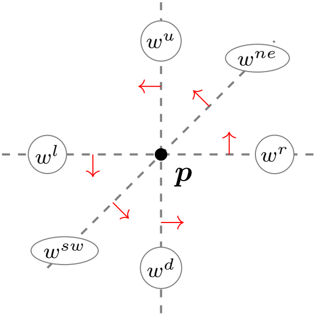

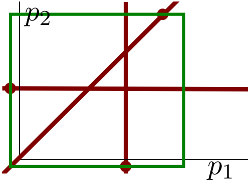

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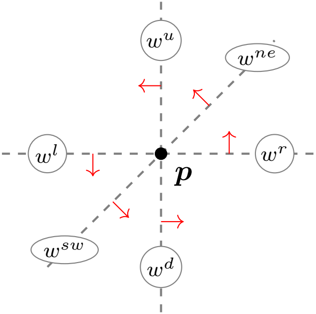

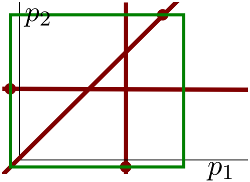

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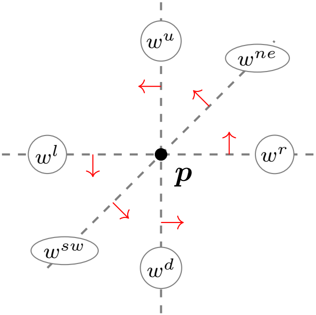

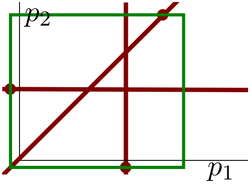

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- on an upper boundary, add a bid:
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More Detail: Key Lemma

1. Given a diagonal line H, and $\boldsymbol{p} \in H$:
   - if $\boldsymbol{p}$ lies on the upper boundary, $w_v^{sw} = w_{\mathcal{B}}^{sw}$
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3. Vertical version analogous to horizontal.

More detail: $\mathcal{H}_v = \mathcal{H}_{\mathcal{B}}$

Fix a horizontal line $H \subseteq \mathcal{H}_v$.

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Trace left along H from F. See H contains a point $\boldsymbol{p}$ such that either

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So $H \subseteq \mathcal{H}_{\mathcal{B}}$.

Converse and other slopes identical.

More Detail: $w_v = w_{\mathcal{B}}$

Fix a horizontal line $H \subseteq \mathcal{H}_v = \mathcal{H}_{\mathcal{B}}$.

H composed of the same facets in both $\mathcal{H}_v$ and $\mathcal{H}_{\mathcal{B}}$.

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So weights of all facets in H are the same.

Argument analogous for vertical, diagonal.

Sketch Proof of Representation in 2-D case: Reminder

Given a valuation v for substitutes

1. Extend all lines in the LIP $\mathcal{L}_v$ to create a "HIP" $\mathcal{H}_v$, a union of doubly-infinite lines: "hyperplanes of indifference prices".
2. Define "Bounding box" containing all vertices of $\mathcal{H}_v$. Write $[\mathcal{H}_v]$ to also include edges of this box.
3. Create the bid list $\mathcal{B}$, as just detailed.
   - Additional bid for only each good i, multiplicity $D_v(\mathbf{p})_i$ when $p_i \gg 0$.
4. See that, on each good, demands match for high enough prices.
5. See $\mathcal{H}_v = \mathcal{H}_{\mathcal{B}}$
6. See $w_v = w_{\mathcal{B}}$
7. $D_v = D_{\mathcal{B}}$, for each good, by "facet normal times weight equals change in demand".

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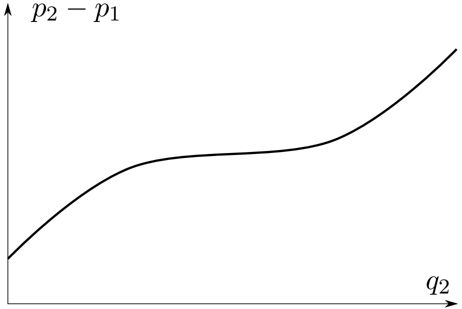
Summary

Seller preferences: Method 1

- Seller has preferences e.g. $q_1 + q_2$ is constant; q_2 as a function of $p_2 - p_1$. This is the 'supply curve'.
- For a set of relevant values of (q_1, q_2), find (minimum) prices (p_1, p_2) such that (q_1, q_2) is demanded.
- This allows us to derive a 'demand curve'.
- Intersect supply and demand to find the equilibrium.



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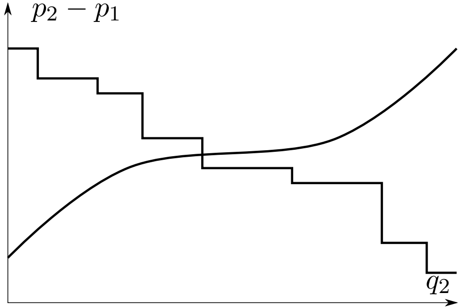

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Advantages:

- People in business and central bankers understand.
- Can use for a wide range of seller preferences (not necessarily strong substitute).

Disadvantage:

- Could be ad-hoc and computationally inefficient.

Seller preferences: Method 2

Suppose the seller has strong substitute preferences also.
That is, seller has a valuation $v_S : A^S \to \mathbb{R}$, where typically $A^S \subsetneq \mathbb{Z}^n_-$.
This valuation is concave and of the strong substitute demand type.

Definition

There exists *competitive equilibrium* between this seller and buyers with aggregate valuation V if there exists $\mathbf{p}$ such that $\mathbf{0} \in D_{v_s}(\mathbf{p}) + D_V(\mathbf{p})$.

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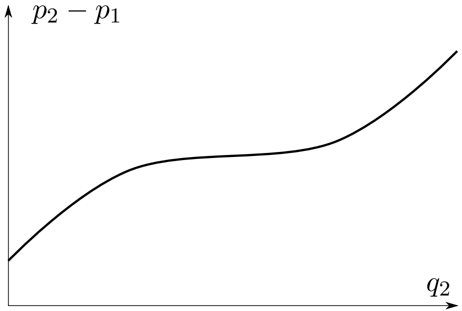
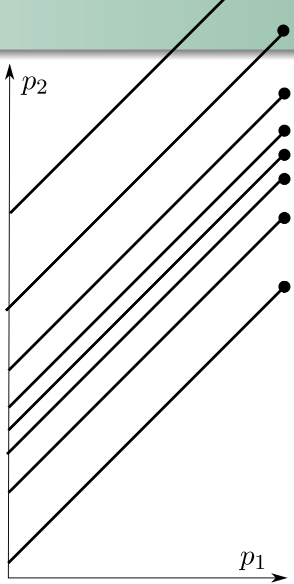
Easy to avoid negative bundles:

- Add a large enough constant bundle $\mathbf{y}$ to every seller demand
- Let $\mathbf{y}$ be the supply available in the auction

Seller preferences: Method 2



Auctioneer's Supply Curve



Corresponding LIP showing Auctioneer's "demand"

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