Online Learning and Collusion in Multi-unit Auctions

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SLMath Randomization, Neutrality, and Fairness Workshop

Joint with

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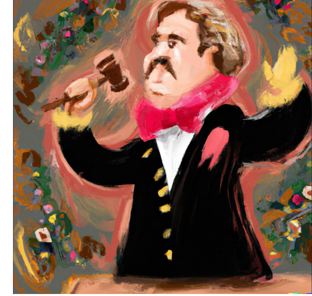
and **Yanjun Han** (New York University)

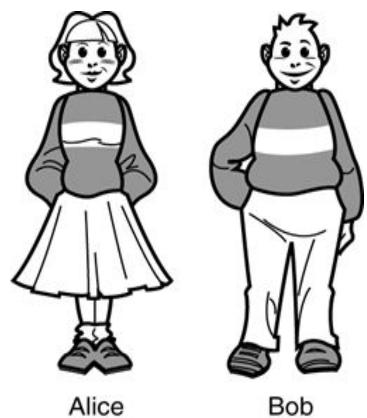
Multi-unit Auction

Seller brings multiple identical units of a good (e.g. chairs)

Set of buyers have money and may be interested in purchasing the goods.



Buyers



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Diminishing marginal returns: $v_{i,1} \geq \cdots \geq v_{i,K}$.

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Valuations: $v_{Alice} = (5, 2)$ and $v_{Bob} = (4, 1)$.

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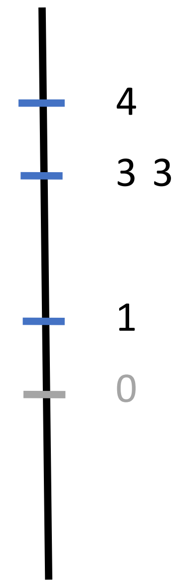

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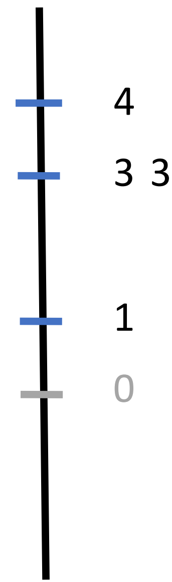

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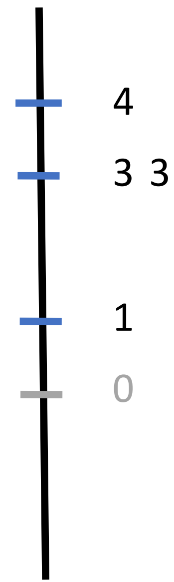

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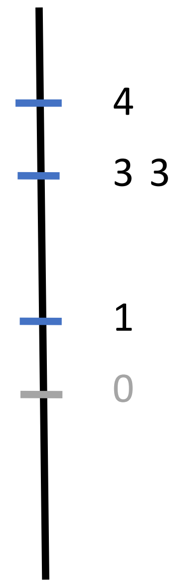

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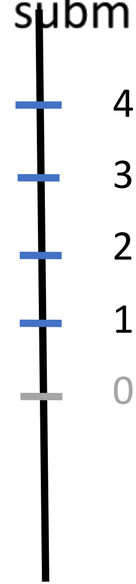

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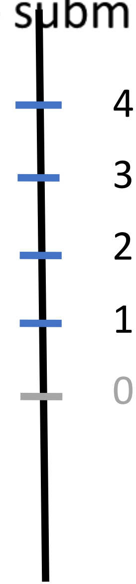

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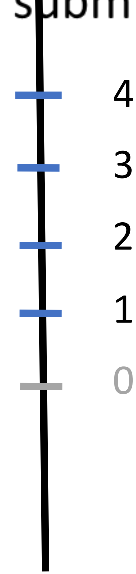

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Auctioneer can decide to hide some information from a player

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- The format used in practice is based on the

Related Work

Large body of literature on multi-unit auctions (e.g. Maskin-Riley-Hahn ’89; Engelbrecht-Kahn ‘98; Borgs-Chayes-Immorlica-Mahdian-Saberi ’05; J. Hartline, A. Karlin, D. Kempe, C. Kenyon, and F. McSherry ‘05; Dobzinski-Nisan ’07; Dobzinski-Lavi-Nisan ’12; Feldman-Fiat-Leonardi-Sankowski ’12; Goel-Mirrokni-Leme ‘13; Ausubel-Cramton-Pycia-Rostek-Weretka ’14).

Repeated multi-unit auctions: allocating carbon licenses (Cramton-Kerr ’02; ads in online settings (Balseiro-Besbes-Weintraub 15); US treasury notes (Markakis-Telelis 15).

Related Work

Repeated auctions and quality of outcomes reached (Lucier-Borodin ’10; Bhawalkar and Roughgarden ’11; Daskalakis-Syrganis 16; Nedelec-Calauzènes-Karoui-Perchet ‘22)

Outline

- Offline Setting
- Online Setting
- Collusion
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The Offline Problem

Input:

- a player i with valuation v_i and
- a history $H_{-i} = (b^1_{-i}, b^2_{-i}, \ldots, b^T_{-i})$ containing the bids of the other players in rounds 1 through T.

The offline problem is: What is the best response (bid vector) for player i given the historical data? That is:

The Offline Problem: (K+1)-st price example

- Suppose $K = 2$ and Alice's valuation is $v_{Alice} = (3.1, 2.99)$. Suppose $T = 2$ and $n = 3$. The history of bids of the other players (say Bob and Carol) are depicted below.

4

3

2

1.5, 1.5

1, 1

1

0

0

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4

3.1

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- Suppose $K = 2$ and Alice's valuation is $v_{Alice} = (3.1, 2.99)$. Suppose $T = 2$ and $n = 3$. The history of bids of Bob and Carol are depicted below. If Alice bid $b = (3.1, 2)$:

4

3.1 3.1

3

2 2 2

1.5, 1.5

1, 1 1

0 0

The Offline Problem: (K+1)-st price example

- Suppose $K = 2$ and Alice's valuation is $v_{Alice} = (3.1, 2.99)$. Suppose $T = 2$ and $n = 3$. The history of bids of Bob and Carol are depicted below. If Alice bid $b = (3.1, 2)$: she still gets 1 unit each round but pays only 2 each round, with utility $3.1 - 2 = 1.1 > 0.11$ each round (better than before)

4

3.1

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2

2

2

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1, 1

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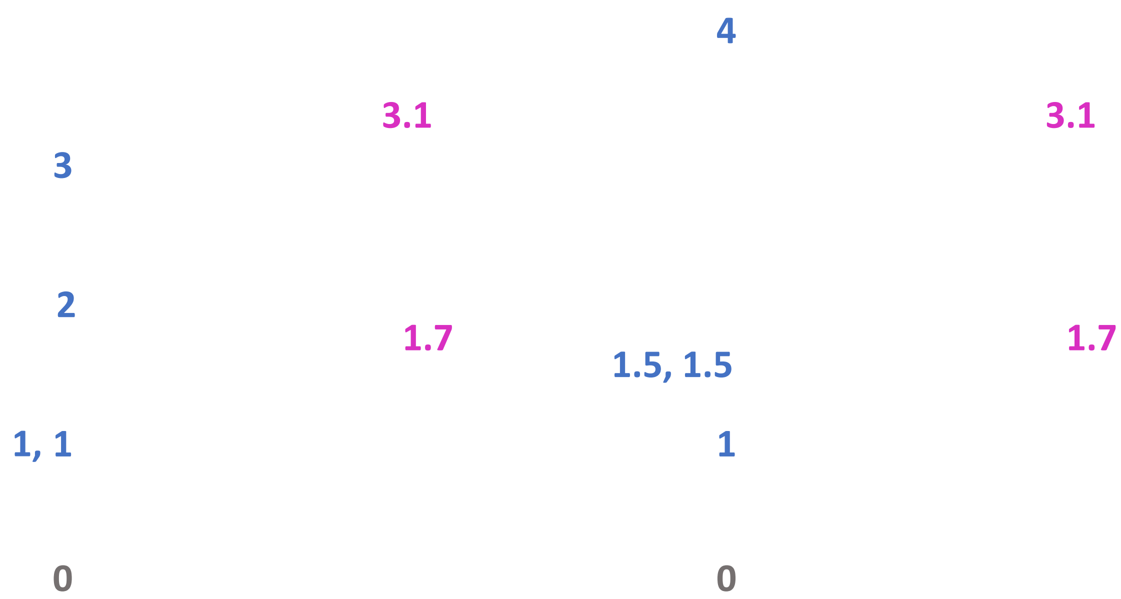

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Another possible bid vector of Alice is $b = (3.1, 1.7)$: then in round 1 Alice gets 0 units and pays 0; in round 2, Alice gets 1 unit and pays 1.7.

4

3.1 3.1

3

2

1.7 1.7

1.5, 1.5

1, 1 1

0 0

The Offline Problem

- For now we can assume the bids are restricted to discrete domain

$$D = \{\ell \cdot \epsilon \mid \ell \in N\}.$$

Observation: There is an optimum bid vector for player i in the following set of “candidate” bids:

$$S_i = \{0\} \cup \{b_{j,k}^{t} \mid j \in [n] \setminus \{i\}, k \in [K]\} \cup$$

Algorithm

- **Theorem:** There is a polynomial time algorithm for the offline problem.

Input: Player i with valuation v_i and history $H_{-i} = (b^1_{-i}, b^2_{-i}, \dots, b^T_{-i})$ with the bids of other players in rounds 1 through T.

Output: Bid vector b_i^* with $b_i^* = \arg\max_{\beta} \sum_{t=1}^{T} u_i(\beta, b^t_{-i})$.

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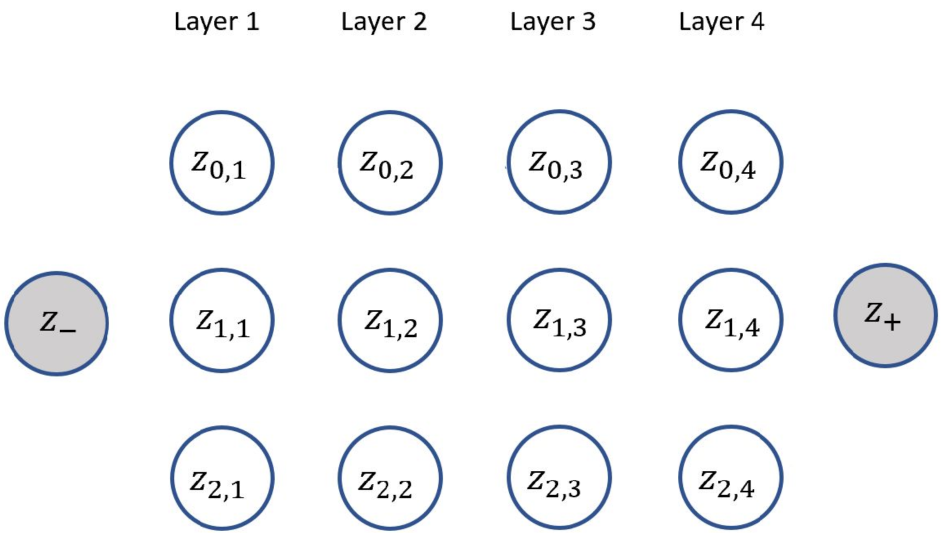
DAG G_i for the offline problem of player i

- **Vertices:** Create a vertex $z_{s,j}$ for each $s \in S_i$ and $j \in [K]$. We say vertex $z_{s,j}$ is in layer j. Add source z_- and sink z_+.

Example: $K = 4$ units;
Set of possible bid values ("candidate" bids) $S_i = \{0, 1, 2\}$.



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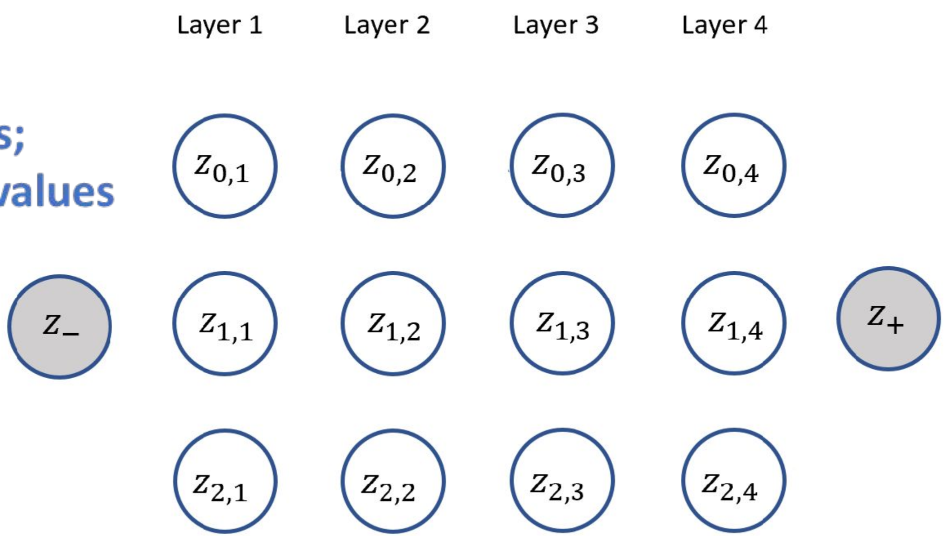

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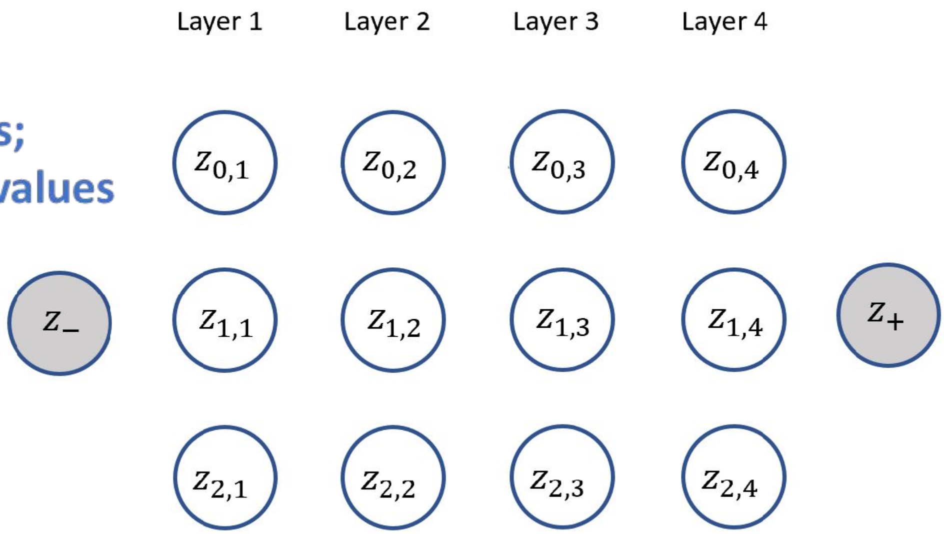

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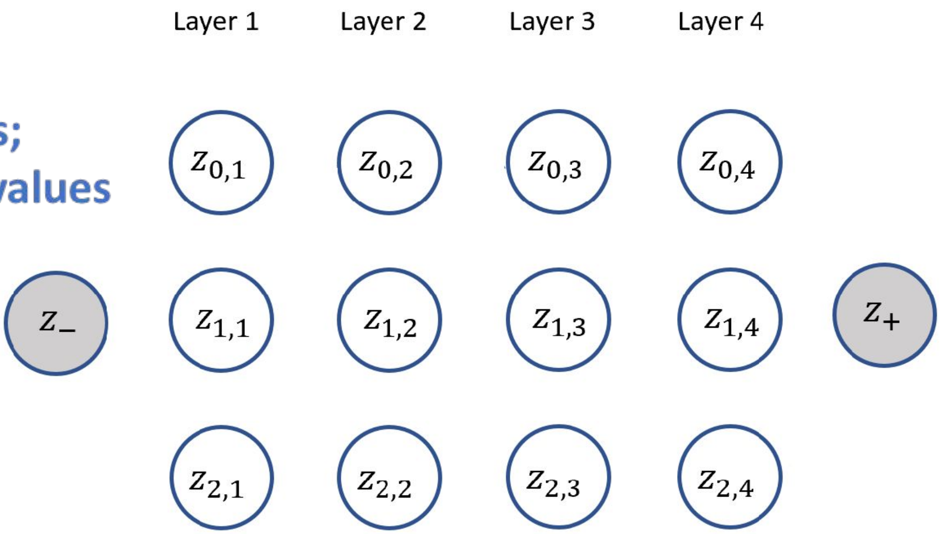

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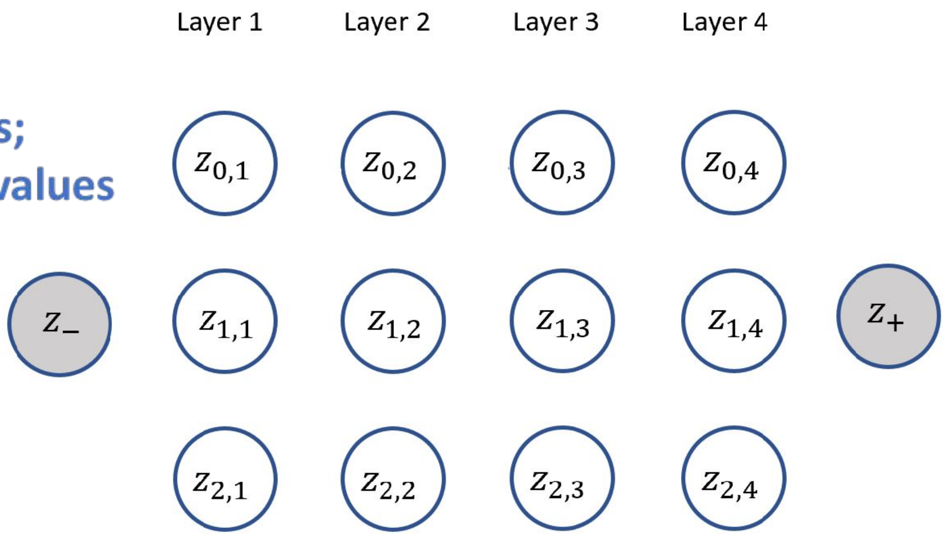

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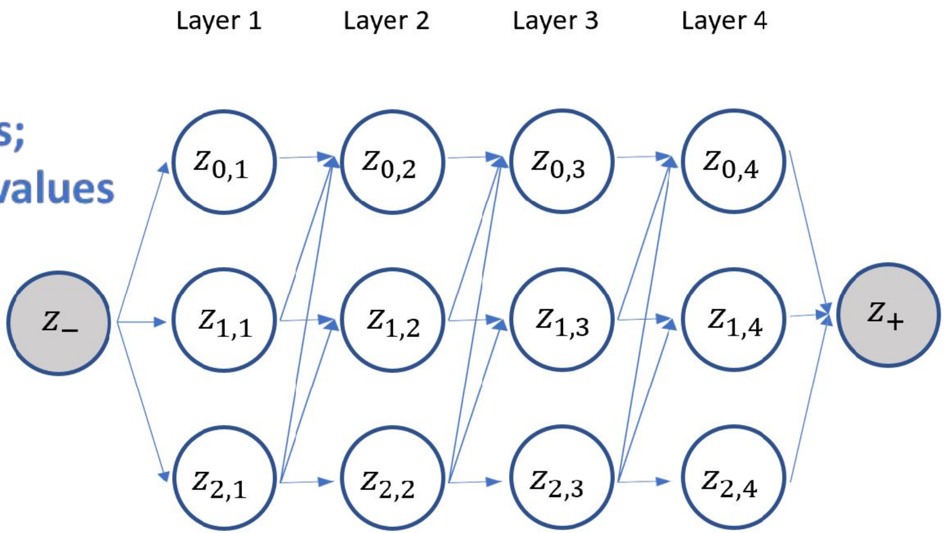

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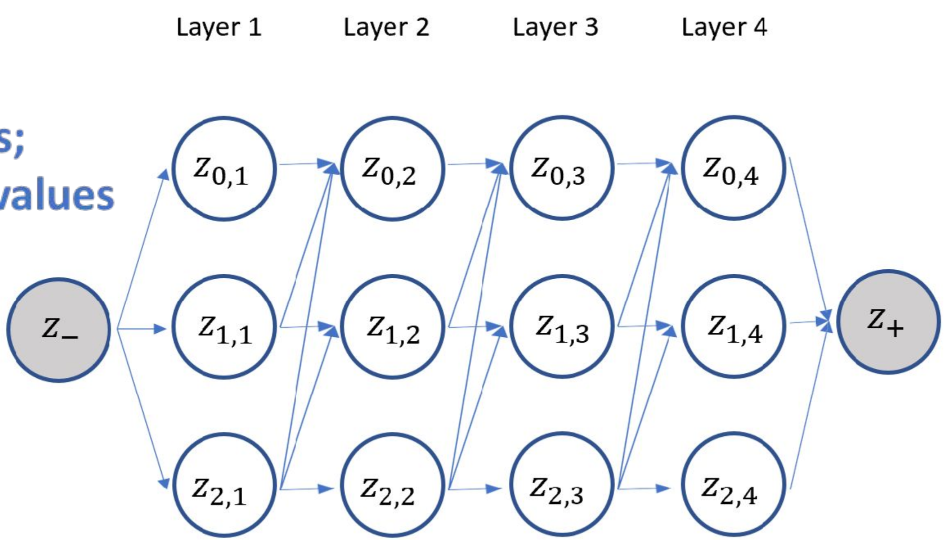
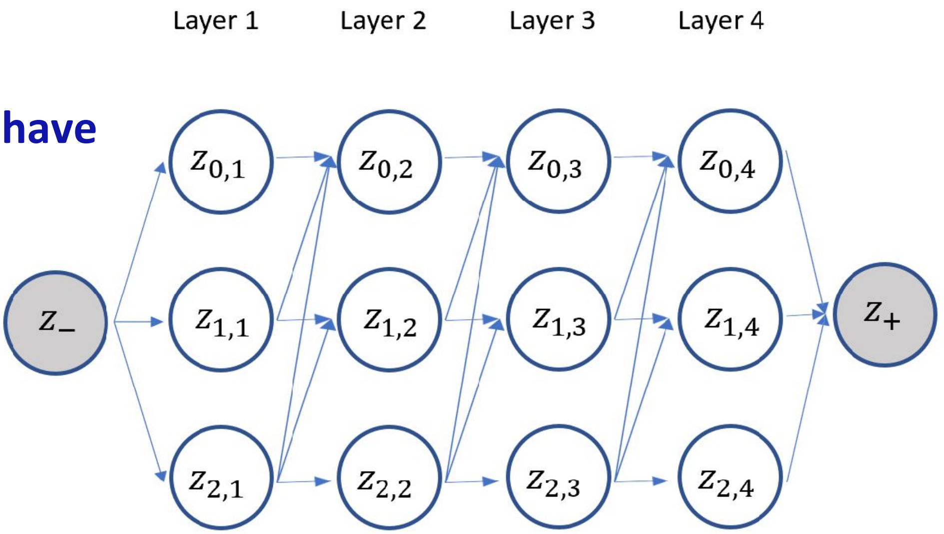


DAG G_i for the offline problem of player i

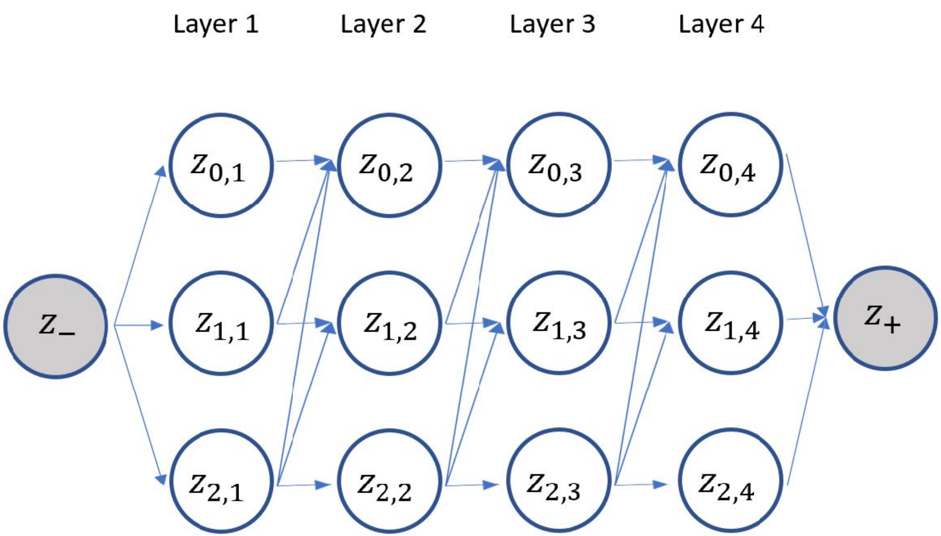
- **Edges:** For each $j \in [K-1]$ and pair of bids $r, s \in S_i$ with $r \geq s$, create directed edge from vertex $z_{r,j}$ to $z_{s,j+1}$. Also add edges from source z_- to each vertex in layer 1 and from each vertex in layer K to the sink z_+.

Note: the nodes do not have equal degrees.



DAG G_i for the offline problem of player i

- **Weights:** For each edge $\mathrm{e} = (\mathrm{z}_{\mathrm{r,j}}, \mathrm{z}_{\mathrm{s,j+1}})$ or $\mathrm{e} = \left(\mathrm{z}_{\mathrm{r},K}, \mathrm{z}_{+}\right)$, let $\beta = (\beta_1, \dots, \beta_K)$ be a bid vector with $\beta_j = r$ and $\beta_{j+1} = s$.



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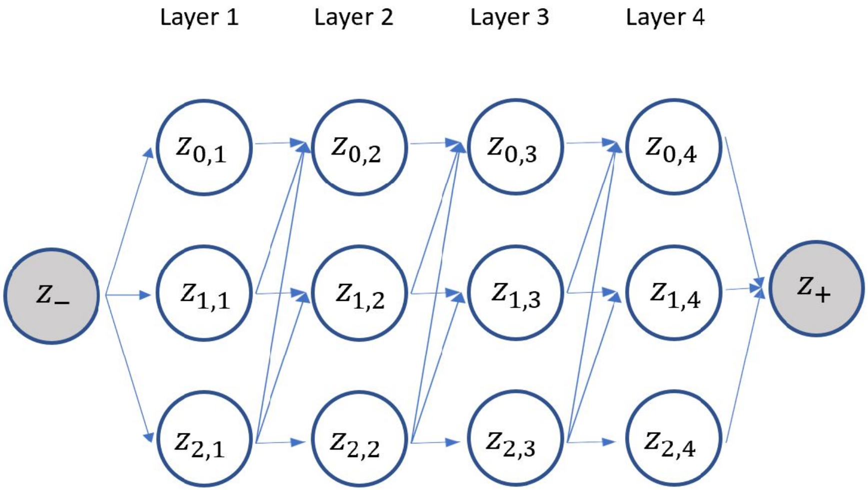

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Formally

$$w_e = \sum_{t=1}^{T} 1\{x_i(\beta, b_{-i}^t) \geq j\}(v_{i,j} - r) +$$

[illegible]

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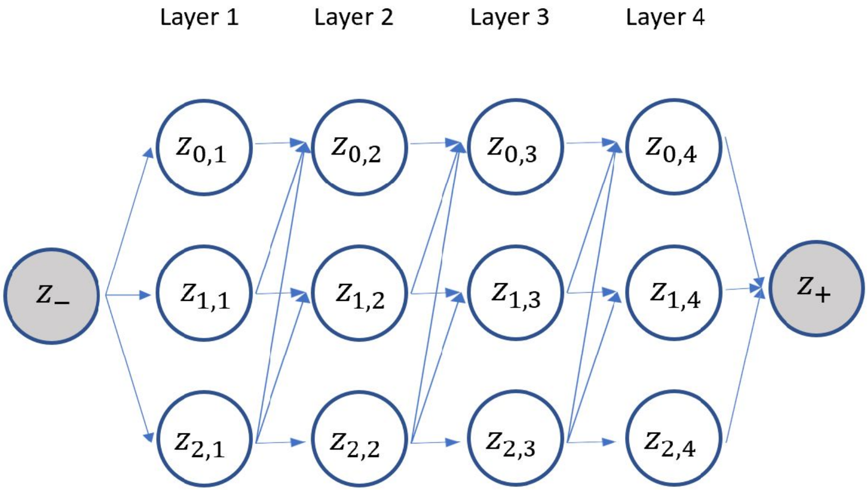

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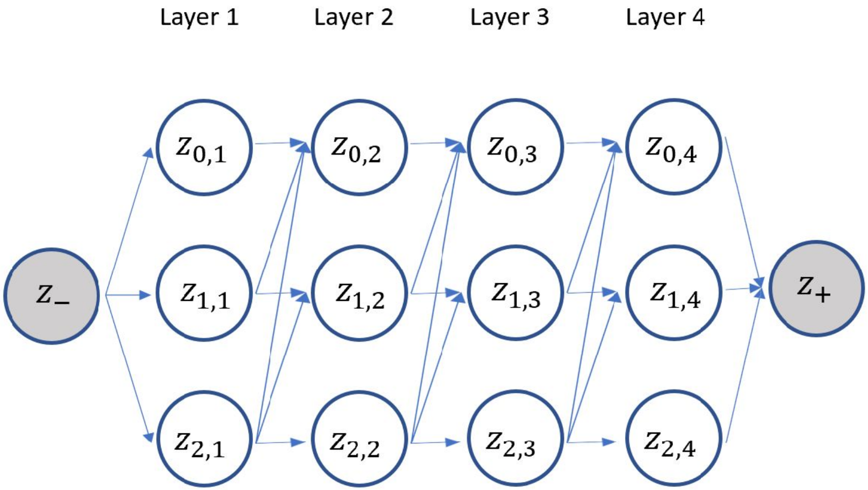

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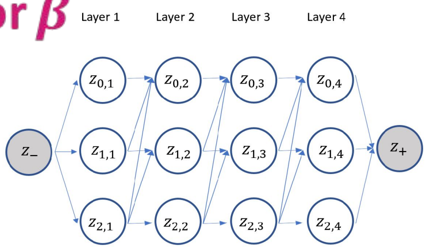
The edge weight depends on entire bid vector β



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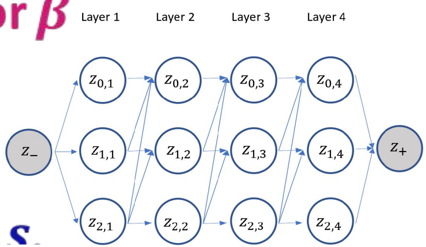

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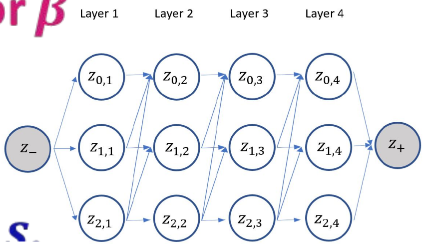

However, the same edge weight is obtained for any bid vector $\boldsymbol{\beta}$ with $\boldsymbol{\beta_j = r}$ and $\boldsymbol{\beta_{j+1} = s}$.

Algorithm

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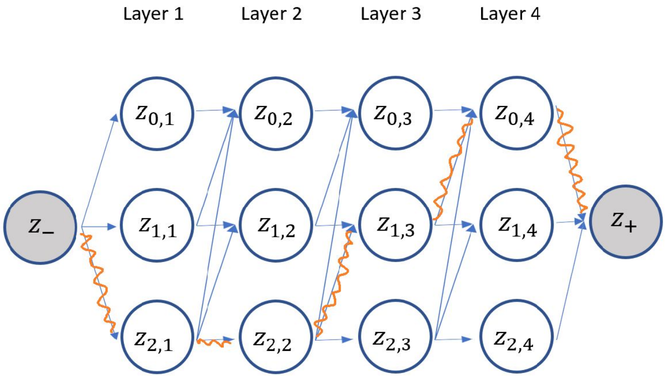
Compute a max weight path in G_i and output the corresponding bid vector.



Outline

- Offline Setting
- Online Setting
- Collusion
- Discussion

Recall the Online Setting

In each round $t = 1, 2, \ldots,$ the next steps take place:

The auctioneer announces K units for sale.

Each player i privately submits bids $b_i^t = \left(b_{i,1}^t, \ldots, b_{i,K}^t\right)$, where $b_{i,j}^t$ is player i's bid for a j-th unit at time t.

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Feedback: Full information – All the bids b^t are public knowledge

The Online Setting – Strategies

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$\boldsymbol{\pi_i} = (\boldsymbol{\pi_i^1}, \dots, \boldsymbol{\pi_i^T})$: the overall strategy of player i over the time

The Online Setting – Regret

- Given a bidding strategy $\pi_i = \left(\pi_i^1, \ldots, \pi_i^T\right)$ of player i, the **regret** of the player is defined with respect to a history H_{-i}^t of bids by other players:

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$$- \sum_{t=1}^{T} \mathbb{E}_{\mathbf{b}_i^t \sim \pi_i^t(H_i^{t-1})} \left[\sum_{j=1}^{K} \left(v_{i,j} - p(\mathbf{b}^t)\right) \cdot \mathbb{1}_{\{x_i(\mathbf{b}^t) \geq j\}} \right]$$

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Cumulative utility of player i when using bid vector β in each round while the others have bid profile b_{-i}^t in each round t

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Expected cumulative utility of player i when using mixed strategy π_i in each round (which is only allowed to access the history up to end of the previous round when prescribing what to play)**; the other players have bid profile b_{-i}^t in each round t.**

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For the purpose of giving player i a bidding algorithm, we think of the other players as adversarial, and aim to achieve small regret **regardless of H_{-i}^t.**

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That is, for each edge $e = (z_{r,j}, z_{s,j+1})$ or $e = (z_{r,K}, z_+)$, let $\beta = (\beta_1, \dots, \beta_K)$ be a bid vector with $\beta_j = r$ and $\beta_{j+1} = r$ (where

$$w^t(e) = \mathbb{1}_{\{x_i(\boldsymbol{\beta},\mathbf{b}^t_{-i}) \geq j\}} \left(v_{i,j} - r\right) + j\left[\mathbb{1}_{\{x_i(\boldsymbol{\beta},\mathbf{b}^t_{-i}) > j\}} \left(r - s\right) + \mathbb{1}_{\{x_i(\mathbf{h}^t) = j\}} \left(r - p(\boldsymbol{\beta}, \mathbf{b}^t_{-i})\right)\right]$$

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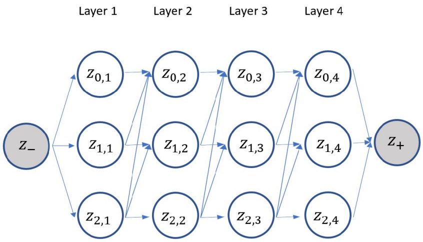
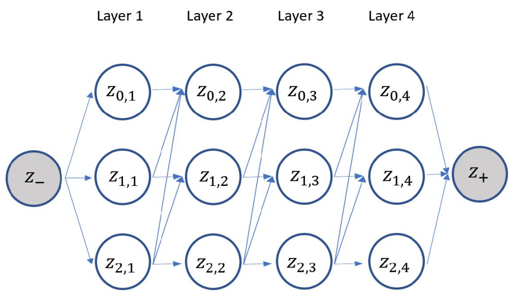


Create a set of **experts**, each corresponding to a path from source to sink in the DAG G^t and run MWU.

The Online Setting – Full Information Feedback

- **Bidding algorithm.** At each time $t \in [T]$, maintain a probability distribution P^t over all the possible paths from the source to the sink in G^t and then sample a path b_i^t



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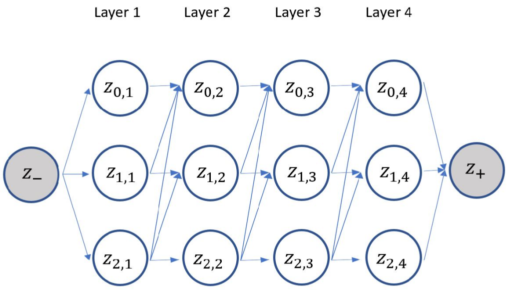
- **Bidding algorithm.** At each time $t \in [T]$, maintain a probability distribution P^t over all the possible paths from the source to the sink in G^t, and then sample a path b_i^t, where P^t is defined as:



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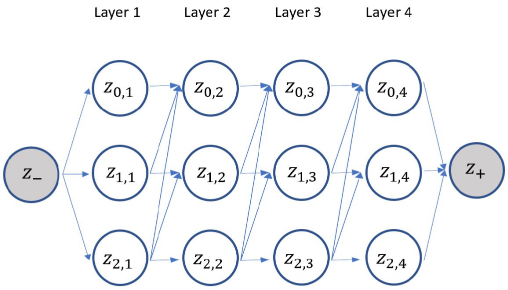

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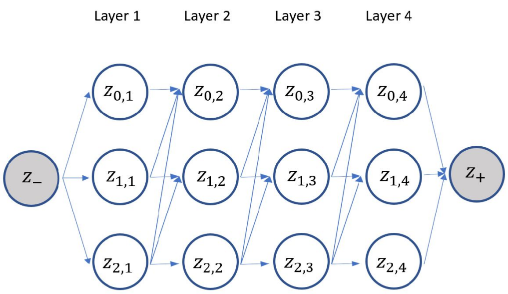

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- For $t \geq 2$, recursively define for all paths $\mathfrak{p}$

$$P^t(\mathfrak{p}) = \frac{P^{t-1}(\mathfrak{p}) \exp(\eta \sum_{e \in \mathfrak{p}} w^{t-1}(e))}{\sum_{\mathfrak{q}} P^{t-1}(\mathfrak{q}) \exp(\eta \sum_{e \in \mathfrak{q}} w^{t-1}(e))}$$

The Online Setting – Full Information Feedback

- **Note:** The bidding algorithm described is also known as Hedge:

- N experts, each expert $p \in [N]$ is a path from source to sink
- learning rate η, time horizon T, max reward $L = K\, v_{i,1}$
- initial distribution σ over the experts: $\sigma_p = P^1(p) \geq 1/\left\lceil \frac{v_{i,1}}{\epsilon} \right\rceil^K$

Forecaster starts with initial distribution σ, then at each step predicts according to an expert drawn from distribution, observes the utility (i.e. how good it was to listen to each expert) and updates probability of choosing each expert next time (less likely to listen to experts that gave bad advice so far).

The Online Setting – Full Information Feedback

- The regret of the learner is at most $\frac{1}{\eta} \cdot \max_{p \in [N]} \log\left(\frac{1}{\sigma_p}\right) + \frac{T L^2 \eta}{8}$.

Setting **discretization level** to $\epsilon = v_{i,1}\sqrt{K/T}$ and **learning rate** to

$\eta = \frac{\sqrt{\log T}}{v_{i,1}\sqrt{KT}}$ gives **regret upper bound** of $O\left(v_{i,1} \cdot \sqrt{TK^3 \log(T)}\right)$

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The Online Setting – Full Information Feedback

Polynomial time implementation.

For the efficient implementation, we follow the work of Takimoto-Warmuth '03 on path kernels. Simulate MWU using an "indirect" algorithm that maintains only probability per edge rather than per each path.

The Online Setting – Full Information Feedback

- **Polynomial time implementation.** We decompose the probability $P^t(p)$ of choosing path p in the form $P^t(p) = \prod_{e \in p} \phi^t(e)$, for an appropriate choice of ϕ^t. For each vertex u, $\phi^t(u, \cdot)$ is a probability distribution over the outneighbors of u.

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Regret lower bound

- **For both full information and bandit feedback.**

Regret lower bound

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Regret lower bound

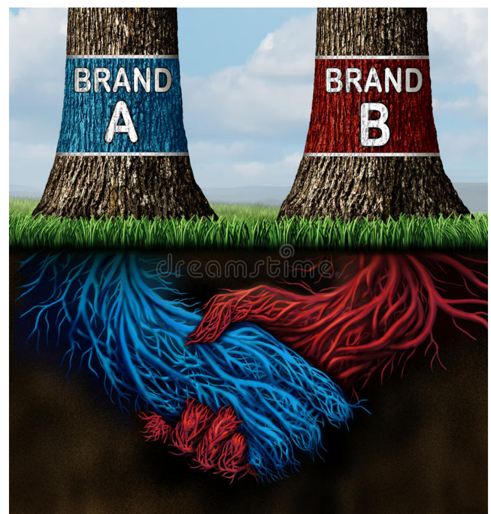
- **Sketch (cont.):** $K = 2k$ and $v_{ij} = 1$ for all $j \in [K]$.

Outline

- Offline Setting
- Online Setting
- **Collusion**
- Discussion

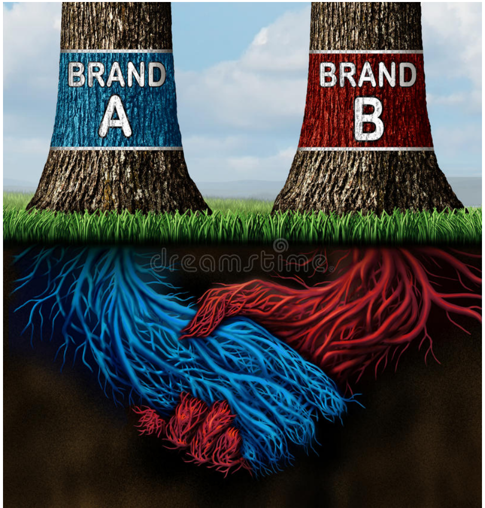
A few words about quality of equilibria

Bidders in an auction can collude to improve their utilities together (e.g. bid low to keep the price small), pay each other for favors. Repeated auctions allow bidders to use bid rotation schemes as the history of bids can serve as a communication device (e.g. Aoyagi ’03).



A few words about quality of equilibria

Collusion can emerge naturally even when the agents are not trying to conspire but use q-learning algorithms to update their strategies (e.g. Calvano-Calzolari-Denicolo-Pastorello ’21, on algorithmic collusion in markets such as Cournot oligopolies).



A few words about quality of equilibria

- Consider the game between the players (bidders).

A few words about quality of equilibria

- **From prior work** - see e.g. Kremer-Nyborg ’04; Ausubel-

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H

K bids of value $H \gg max_{i,j} v_{ij}$
(some very large value)
All other bids zero.

K bids

True valuations,
in (0, H)

Remaining
bids

0

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- The winners pay zero.

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K bids	H	K bids of value $H \gg max_{i,j} v_{ij}$ (some very large value) All other bids zero.
True valuations, in (0, H)		**Nobody can deviate and improve:**
Remaining bids	0	

- The winners pay zero.
- The losers would have to bid $\geq H$

But then they would pay

$$H \gg max_{i,j} v_{ij}$$

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In contrast, no such equilibrium exists in the K-th price auction.

E.g. this profile is not a Nash equilibrium in the K-th price auction

0

A few words about quality of equilibria

- **From prior work** - see e.g. Kremer-Nyborg '04; Ausubel-

In contrast, no such equilibrium exists in the K-th price auction. **Why?** If the price is 0, it means the K-th highest bid is 0 and all losing bids are 0.

H

K-1 bids

All other bids

0

A few words about quality of equilibria

- **From prior work** - see e.g. Kremer-Nyborg ’04; Ausubel-

In contrast, no such equilibrium exists in the K-th price auction. **Why?** If the price is 0, it means the K-th highest bid is 0 and all losing bids are 0.

When players are hungry there is a player i with value $v_{i1} \in (0, H)$ who can bid a small $\epsilon > 0$ and get a unit => its utility will be $v_{i1} - \epsilon > 0$.

Core of the game between the bidders

Consider the game between the players (bidders) using the core solution concept.

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The core of auctions has been studied extensively, usually allowing the auctioneer to collude together with the players (see, e.g., Ausubel-Milgrom ’02, Krishna-Maenner ’01).

Core of the game between the bidders

Consider the game between the players (bidders) using the core solution concept.

The core of auctions has been studied extensively, usually allowing the auctioneer to collude together with the players (see, e.g., Ausubel-Milgrom ’02, Krishna-Maenner ’01).

We will assume the auctioneer chooses the auction format and then does not take further actions except to run it. The bidders may coordinate their actions.

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- A strategy profile $b = (b_1, \ldots, b_n)$ is (core) stable if no group of

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- **Theorem.** Consider K units and $n > K$ hungry players. The core

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Recall hungry players means $v_{ij} > 0 \ \forall \ i, j$

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- **Theorem.** Consider K units and $n > K$ hungry players. The core

Same construction as for Nash equilibrium of the $(K+1)$-st auction.

$$H \gg \max_{ij} v_{ij}$$

This bid profile is stable (no coalition can deviate and improve).

H

K bids

All other bids

0

Core with transfers

- Consider strategy profiles of the form (b, t), where $b\ =$

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Welfare = sum of valuations
It really means the players
with highest valuations win.
In this setting these can be seen
as the players with most money.

Core with transfers

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Outline

- Offline Setting
- Online Setting
- Collusion
- **Discussion**

Discussion

- The $(K+1)$-st price auction is used for selling carbon licenses

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Recall the Online Setting

In each round $t = 1, 2, \ldots,$ the next steps take place:

The auctioneer announces K units for sale.

Each player i privately submits bids $b_i^t = \left(b_{i,1}^t, \ldots, b_{i,K}^t\right)$, where $b_{i,j}^t$ is player i's bid for a j-th unit at time t.

Auctioneer runs the auction with bids $b^t = \left(b_1^t, \ldots, b_n^t\right)$ and reveals information (feedback) about the outcome to the players.

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Feedback: Full information – All the bids b^t are public knowledge