Characterizing general top trading cycles mechanisms

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The Point of Departure

The classical housing market model by Shapley and Scarf (1974): "Cores and Indivisibilities," *Journal of Mathematical Economics* 1.

- n agents: $N = \{1, \ldots, n\}$;
- n indivisible and heterogeneous items (e.g., houses): $H = \{h_1, \ldots, h_n\}$;
- endowments: agent $i \in N$ owns house $h_i \in H$;
- preferences: agent $i \in N$ has weak preferences R_i over houses H (notation P_i, I_i, and R_i); and
- agents can trade their houses with no transfers of money.

In their seminal article, Shapley and Scarf model "trading in commodities that are inherently indivisible" as NTU games.

Notation for such a game is V.

- A preference profile $R = (R_1, \ldots, R_n)$;
- $\mathcal{W}$ denotes the set of all weak preference profiles and
- $\mathcal{S}$ denotes the set of all strict preference profiles.
- A Shapley-Scarf housing market (N, H, R) will for short be denoted by R.

  Thus, the sets $\mathcal{W}$ / $\mathcal{S}$ also denote the sets of Shapley-Scarf housing markets with weak / strict preferences.
- An allocation $a = (a_1, \ldots, a_n)$ is a feasible (re)assignment of houses to agents;
- a_i denotes the allotment of agent i
- and for a coalition $S \subseteq N$, $a(S) = \bigcup_{i \in S} a_i$.

(Weak) Core Allocations (Shapley and Scarf, 1974)

A coalition $S \subseteq N$ strongly blocks allocation a if there exists an allocation b such that:

a. $b(S) = h(S)$ and

b. for all $i \in S$, $b_i \, P_i \, a_i$.

Allocation a is a (weak) core allocation ($a \in C(R)$) if it is not strongly blocked by any coalition.

Strong Core Allocations (Roth and Postlewaite, 1977)

A coalition $S \subseteq N$ weakly blocks allocation a if there exists an allocation b such that a. and

b'. for all $i \in S$, $b_i \, R_i \, a_i$ and for some $j \in S$, $b_j \, P_j \, a_j$.

Allocation a is a **strong core allocation** ($a \in SC(R)$ if it is not weakly blocked by any coalition.

Weak domination (and the strong / strict core) was considered by

Roth and Postlewaite (1977): "Weak Versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics* 4.

Shapley and Scarf (1974), Section 4

V is a balanced game; hence the market in question has a non-empty core.

"After the proof in sect. 4 had been discovered, **David Gale** pointed out to the authors a simple constructive method of finding competitive prices in this market, and hence a point in its core."

Main Result (Shapley and Scarf, 1974)

For each $R \in \mathcal{W}$, the set of allocations resulting from the TTC algorithm (explained on the next slide) coincides with the set of competitive allocations $CA(R)$ and

$$\emptyset \neq CA(R) \subseteq C(R).$$

Top trading cycles (TTC) algorithm:

(it's golden anniversary is coming up!):

Input. A Shapley-Scarf housing market $R \in \mathcal{S}$ (for $R \in \mathcal{W}$ break ties).

Step 1. Let $N_1 := N$ and $H_1 := H$. We construct a directed graph with the set of nodes $N_1 \cup H_1$.

For each agent $i \in N_1$ we add a directed edge to his most preferred house in H_1. For each directed edge (i, h) we say that agent i points to house h. For each house $h \in H_1$ we add a directed edge to its owner.

A **trading cycle** is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent in a trading cycle the house he points to and remove all trading cycle agents and houses. We define N_2 to be the set of remaining agents and H_2 to be the set of remaining houses and, if $N_2 \neq \emptyset$, we continue with Step 2. Otherwise we stop.

In general at Step t we have the following:

Step t. We construct a directed graph with the set of nodes $N_t \cup H_t$ where $N_t \subseteq N$ is the set of agents that remain after Step $t-1$ and $H_t \subseteq H$ is the set of houses that remain after Step $t-1$.

For each agent $i \in N_t$ we add a directed edge to his most preferred house in H_t. For each house $h \in H_t$ we add a directed edge to its owner.

At least one trading cycle exists and we assign to each agent in a trading cycle the house he points to and remove all trading cycle agents and houses. We define N_{t+1} to be the set of remaining agents and H_{t+1} to be the set of remaining houses and, if $N_{t+1} \neq \emptyset$, we continue with Step $t+1$. Otherwise we stop.

Output. The TTC algorithm terminates when each agent in N is assigned a house in H (it takes at most $|N|$ steps). We denote the obtained allocation by $\mathrm{TTC}(R)$.

Weak Preferences

When preferences are weak, then the strong core may be empty.

The following example is attributed to Jun Wako: $R \in \mathcal{W}$ such that

$$R_1 : h_2,\ h_3,\ h_1;$$

$$R_2 : [h_1,\ h_3],\ h_2;$$

$$R_3 : h_2,\ h_1,\ h_3.$$

Quint and Wako (2004): “On Houseswapping, the Strict Core, Segmentation, and Linear Programming,” *Mathematics of Operations Research* 29 [$O(n^3)$].

A mechanism f assigns to each Shapley-Scarf housing market $R \in \mathcal{S}$ / $\mathcal{W}$ an allocation $f(R)$.

The top trading cycles (TTC) mechanism assigns to each Shapley-Scarf housing market $R \in \mathcal{S}$ / $\mathcal{W}$ the allocation $\mathrm{TTC}(R)$.

For $\mathcal{W}$, the TTC mechanism uses a fixed tie-breaking rule.

Roth (1982): "Incentive Compatibility in a Market with Indivisible Goods," *Economics Letters* 9.

Theorem (Roth, 1982)

Under the TTC mechanism, it is a weakly dominant strategy for each agent to reveal his true preferences.

TTC is strategy-proof, i.e., for each (R_i, R_{-i}) and R'_i,

$$\mathrm{TTC}_i(R_i, R_{-i}) \; R_i \; \mathrm{TTC}_i(R'_i, R_{-i}).$$

Bird (1984): "Group Incentive Compatibility in a Market with Indivisible Goods," *Economics Letters* 14.

Theorem (Bird, 1984)

The TTC mechanism is group incentive compatible, i.e., no group of agents can misrepresent their preferences and be better off.

TTC is weakly group strategy-proof on $\mathcal{W}$, i.e., there does not exist (R_S, R_{-S}) and R'_S such that

$$\text{for each } i \in S,\ \text{TTC}_i(R'_S, R_{-S})\ P_i\ \text{TTC}_i(R_S, R_{-S}).$$

TTC is group strategy-proof on $\mathcal{S}$, i.e., there does not exist (R_S, R_{-S}) and R'_S such that

$$\text{for each } i \in S,\ \text{TTC}_i(R'_S, R_{-S})\ R_i\ \text{TTC}_i(R_S, R_{-S}) \text{ and}$$

$$\text{for some } j \in S,\ \text{TTC}_j(R'_S, R_{-S})\ P_j\ \text{TTC}_j(R_S, R_{-S}).$$

We can easily see that on $\mathcal{W}$ TTC (with fixed tie-breaking) is neither group strategy-proof nor Pareto efficient.

$R_1 : [h_1,\ h_2]; \rightarrow$ after tie-breaking $R_1 : h_1,\ h_2$

$R_2 : h_1,\ h_2.$

Ma (1994): "Strategy-Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory* 23.

Theorem (Ma, 1994)

On $\mathcal{S}$, mechanism f satisfies individual rationality, Pareto efficiency, and (group) strategy-proofness if and only if $f = \mathrm{TTC}$.

On the subset of $\mathcal{W}$ with a non-empty strong core, a correspondence mechanism F satisfies individual rationality, Pareto efficiency, and strategy-proofness if and only if F selects from the strong core.

Alternative proofs of Ma's result are provided in

- Svensson (1999): "Strategy-Proof Allocation of Indivisible Goods," *Social Choice and Welfare* 16.
- Anno (2015): "A Short Proof for the Characterization of the Core in Housing Markets," *Economics Letters* 128.
- Sethuraman (2016): "An Alternative Proof of a Characterization of the TTC mechanism," *Operations Research Letters* 44.

Theorem (Sethuraman, 2016)

On $\mathcal{S}$ there is at most one mechanism that satisfies individual rationality, Pareto efficiency, and strategy-proofness.

Recall that Ma (1994) showed that on the subset of $\mathcal{W}$ with a non-empty strong core,

- the core is essentially single-valued and
- that a correspondence mechanism F satisfies individual rationality, Pareto efficiency, and strategy-proofness if and only if F selects from the strong core.

Sönmez (1999): "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica* 67.

Theorem (Sönmez, 1999)

For generalized indivisible goods allocation problems (no indifferences with endowments, plus domain richness): individual rationality, Pareto efficiency, and strategy-proofness imply essential single-valuedness and selection from the strong core.

The properties used in Ma's (1994) characterization are logically independent.

- Can we weaken or change the properties?
- Are there alternative characterizations with "different" properties?

Takamiya (2001): "Coalition Strategy-Proofness and Monotonicity in Shapley-Scarf housing markets," *Mathematical Social Sciences* 41.

Theorem (Takamiya, 2001)

On $\mathcal{S}$, mechanism f satisfies individual rationality,

- ontoness / unanimity, and
- group strategy-proofness

  [or equivalently, strategy-proofness and non-bossiness];

if and only if $f = \mathrm{TTC}$.

Fujinaka and Wakayama (2018): "Endowments-Swapping-Proof House Allocation," *Games and Economic Behavior* 111.

Weak endowments-swapping-proofness excludes a pair of agents ex-ante swapping their endowments and both being strictly better off by doing so.

Theorem (Fujinaka and Wakayama, 2018)

On $\mathcal{S}$, mechanism f satisfies individual rationality, strategy-proofness, and

- weak endowments-swapping-proofness

if and only if $f = \mathrm{TTC}$.

Reallocation-proofness: no pair of agents can strictly benefit by misreporting preferences and swapping allotments ex post, i.e., for no pair $\{i,j\} \subseteq N$ there are preferences (R'_i, R'_j) such that

$$f_j(R'_i, R'_j, R_{N\setminus\{i,j\}})\; P_i\, f_i(R)$$

and

$$f_i(R'_i, R'_j, R_{N\setminus\{i,j\}})\; P_j\, f_j(R).$$

Preferences-swapping–reallocation-proofness: no pair of agents can strictly benefit by swapping preferences and swapping allotments ex post, i.e., use $R'_i = R_j$ and $R'_j = R_i$ above.

Theorem (Fujinaka and Wakayama, 2018)

On $\mathcal{S}$, mechanism f satisfies individual rationality, strategy-proofness, and

- preferences-swapping–reallocation-proofness (reallocation-proofness)

if and only if $f = \mathrm{TTC}$.

Ekici (2023): "Pair-Efficient Reallocation of Indivisible Objects," *Theoretical Economics*, forthcoming.

Pair-efficiency excludes two agents ex-post swapping their allotments and both being strictly better off by doing so.

Theorem (Ekici, 2023)

On $\mathcal{S}$, mechanism f satisfies individual rationality, strategy-proofness, and

- pair-efficiency

if and only if $f = \mathrm{TTC}$.

Note that all the characterizations that follow Ma's result weaken / replace Pareto efficiency (I am not mentioning some other TTC characterizations, e.g., Miyagawa, 2002).

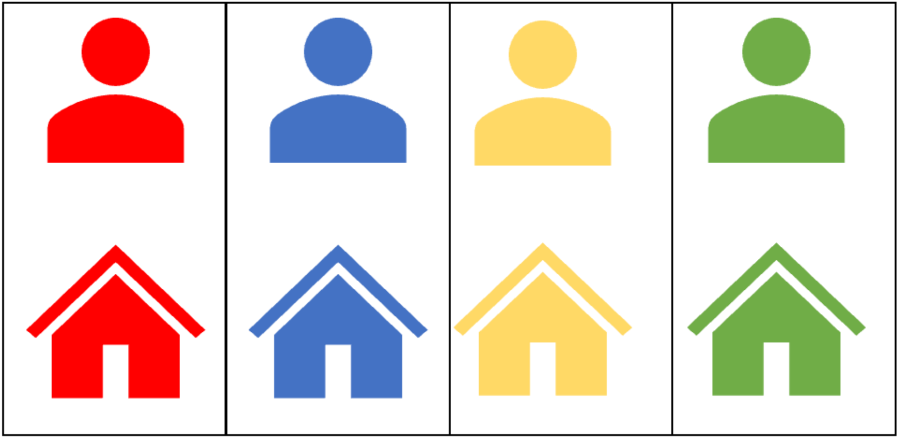
There are many other results emerging when starting from the seminal papers of Shapley and Scarf (1974) and Ma (1994).

- characterization results for weak preferences,
- random mechanisms,
- housing markets with single-peaked preferences,
- kidney exchange.

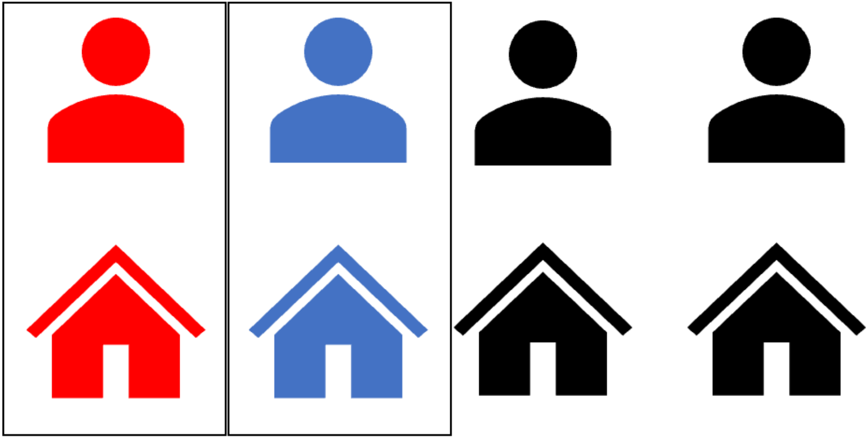
I have worked on:

- house allocating with priorities (school choice),
- multiple-type housing markets,
- farsightedness,
- dynamic recontracting,
- housing markets with externalities, and
- and now, together with **Di Feng**, **object allocation problems with coalitional endowments.**

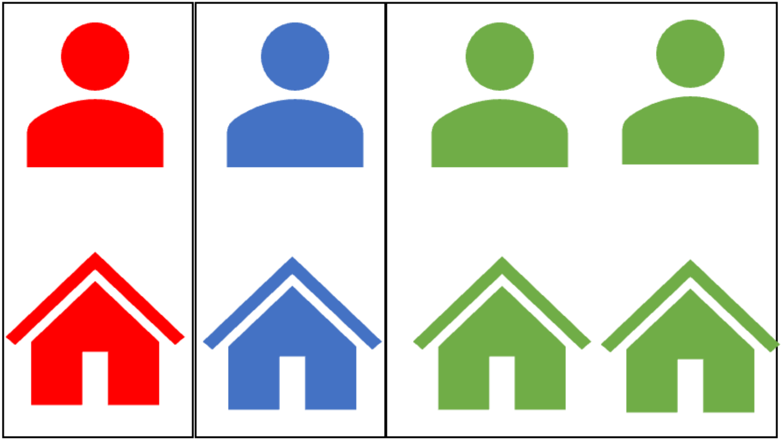
Housing markets (Shapley and Scarf, 1974)



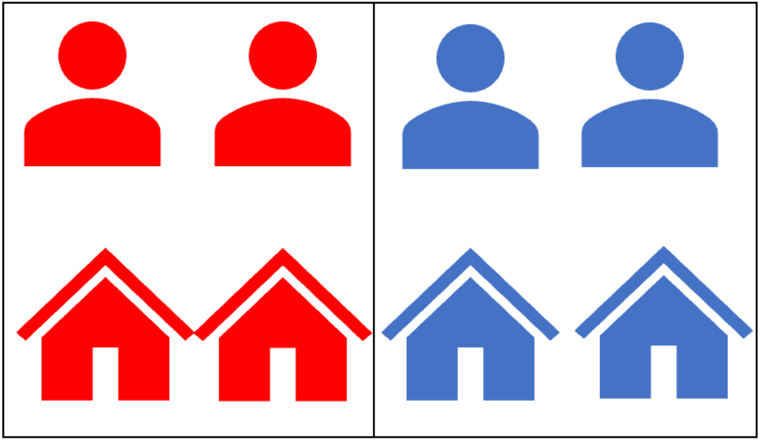
House allocation problems with existing tenants (Abdulkadiroğlu and Sönmez, 1999)



Object allocation problems with coalitional endowments (Park, 2023)



Object allocation problems with coalitional endowments (Park, 2023)



Object allocation problems with coalitional endowments

Changwoo Park (2023), "Core and Incentive Properties for Partitioned-Ownership Economies," Work in Progress, U. of Rochester.

Park introduces the model and he studies (and obtains results for) various core notions.

We still have

- n agents: $N = \{1, \ldots, n\}$;
- n houses: $H = \{h_1, \ldots, h_n\}$;
- strict preference profiles: $(R_1, \ldots, R_n) \in \mathcal{S}$;
- and no monetary transfers.

Object allocation problems with coalitional endowments

The new elements of the model are

- partitioned coalitions: $\mathcal{N} = \{N_1, \ldots, N_\kappa\}$;
- partitioned houses: $\mathcal{H} = \{H_1, \ldots, H_\kappa\}$;
- coalitional endowments: each coalition N_ι owns H_ι; and
- coalitional endowments are balanced: $|N_\iota| = |H_\iota|$.

For instance,

- dormitory allocation with departmental constraints (Sokolov, 2023);
- reallocation in school choice with walk-zone constraints (Kamada and Kojima, 2022).

Sequential priorities-augmented top trading cycles (spaTTC) mechanisms

Park (2023) introduced (Serial) Priorities-Augmented TTC mechanisms. We consider a larger class of mechanisms that inherits features from

- serial and sequential dictatorship mechanisms for house allocation problems;
- you request my house – I get your turn (YRMH-IGYT) mechanisms for house allocation with existing tenants problems (Abdulkadiroğlu and Sönmez, 1999);
- hierarchical exchange mechanisms (Pápai, 2000).

In each coalition, an initial "first dictator" who owns all coalitional endowments exists.

Then, in Step 1, the TTC algorithm is applied to first dictators only. Depending on the trading cycles that form, "second dictators" inherit the remaining coalitional endowments; Steps 2, . . . follow.

An example of a spaTTC mechanism

- $N_1 = \{1, 2, 3\}$ and $N_2 = \{4\}$;
- $O_1 = \{o_1, o_2, o_3\}$ and $O_2 = \{o_4\}$.

Let agent 1 be the first dictator in coalition N_1 and agent 4 is the (trivial) first dictator in coalition N_2.

Define spaTTC mechanism f as follows.

First run the TTC mechanism for first dictators (agents 1 and 4). Then,

- if agent 1 trades with agent 4, the second dictator in coalition N_1 for the remaining objects in O_1 is agent 2;
- otherwise, the second dictator in coalition N_1 for the remaining objects in O_1 is agent 3.

Incentive properties

The following properties of a mechanism f are defined as before.

Group strategy-proofness: no group of agents can misrepresent their preferences and be (weakly) better off.

Reallocation-proofness: no pair of agents can strictly benefit by misreporting preferences and swapping allotments ex post.

Preferences-swapping–reallocation-proofness: no pair of agents can strictly benefit by swapping preferences and swapping allotments ex post.

Coalitional-endowments-lower-bound

The following property is an extension of individual rationality for Shapley-Scarf housing markets to problems with coalitional endowments.

Coalitional-endowments-lower-bound: no agent in a coalition will be worse off than they can be at the coalitional endowment.

Consider preference profile R. Then, for each agent $i \in N_\iota \in \mathcal{N}$, we denote agent i's worst coalitional endowment by $\underline{o}_i \in O_\iota$, i.e., for each $o \in O_\iota$,

$$o \; R_i \; \underline{o}_i.$$

Formally, for each $R \in \mathcal{R}^N$ and each $i \in N$,

$$f_i(R) \; R_i \; \underline{o}_i.$$

Coalitional-endowments-neutrality

Coalitional-endowments-neutrality: an extension of the classical neutrality property with respect to coalitional endowments.

By coalitional-endowments-neutrality, within each coalition, the names of the coalitional endowments do not matter.

General TTC mechanisms that have some of the properties discussed:

- Serial Dictatorships (Svensson, 1999);
- YRMH-IGYT mechanisms (Sönmez and Ünver, 2010);
- Hierarchical Exchange mechanisms (Pápai, 2000).

Characterizations of general TTC mechanisms

Svensson, 1999

For house allocation problems, only serial dictatorships satisfy group strategy-proofness and neutrality.

Sönmez and Ünver, 2010

For house allocation problems with existing tenants, only YRMH-IGYT mechanisms satisfy individual rationality, Pareto efficiency, strategy-proofness, weak neutrality, and weak consistency.

Pápai, 2000

For house allocation problems, only hierarchical exchange mechanisms satisfy Pareto efficiency, group strategy-proofness, and self-enforcing-reallocation-proofness.

Characterizing spaTTC mechanisms

Theorem (Feng and Klaus, 2023)

Mechanism f satisfies

- coalitional-endowments-lower-bound;
- group strategy-proofness;
- preferences-swapping–reallocation-proofness [or reallocation-proofness], and
- coalitional-endowments-neutrality

if and only if it is a spaTTC mechanism.

Thank you for your attention!

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