Smallest denominators and extreme events

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Hot Topics: Interactions between Harmonic Analysis,
Homogeneous Dynamics, and Number Theory
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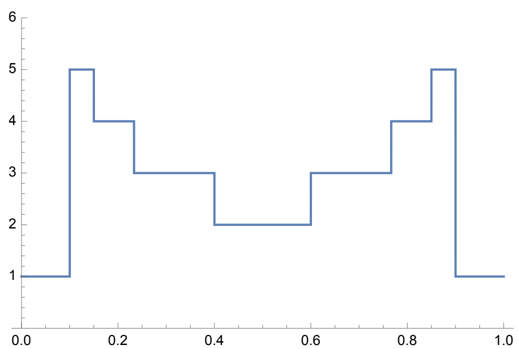
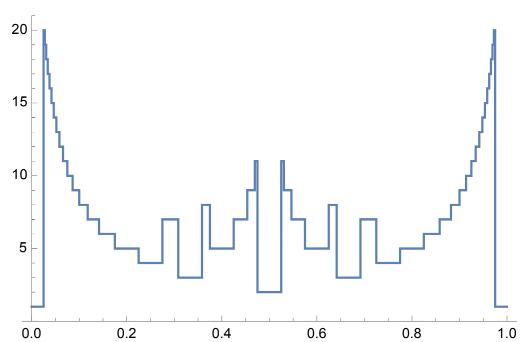
The smallest denominator function

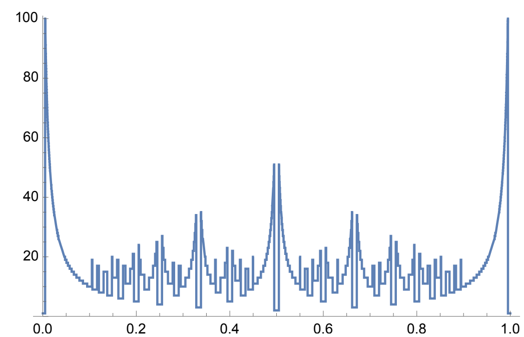
$$q_{\mathrm{min}}(x,\delta) = \min\left\{q \in \mathbb{N} : \exists \tfrac{p}{q} \in \mathbb{Q} \cap \left(x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}\right)\right\}$$



$\delta = 0.2$



$\delta = 0.05$

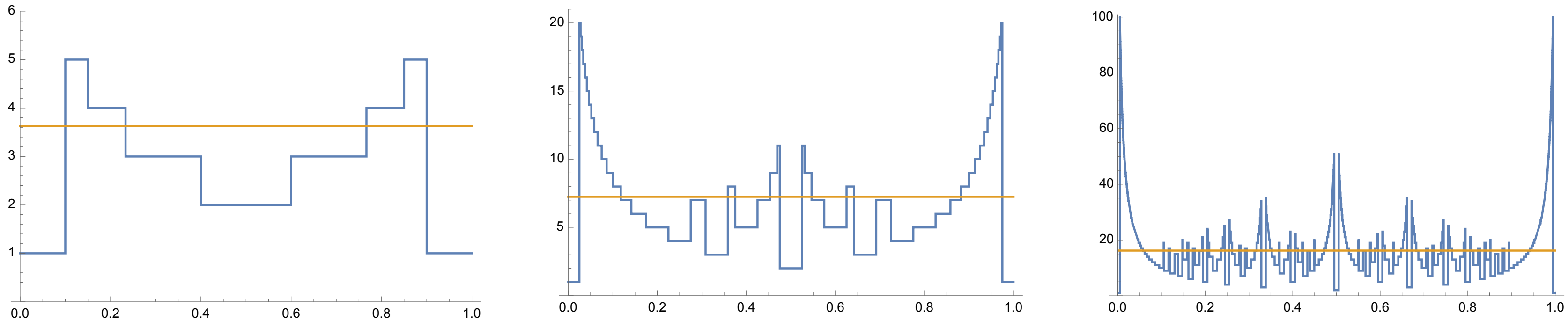


$\delta = 0.01$

Expected value

Theorem A. (Chen & Haynes 2023)

$$\int_0^1 q_{\min}(x,\delta)\,dx = \frac{16}{\pi^2}\,\delta^{-1/2} + O(\log^2 \delta)$$

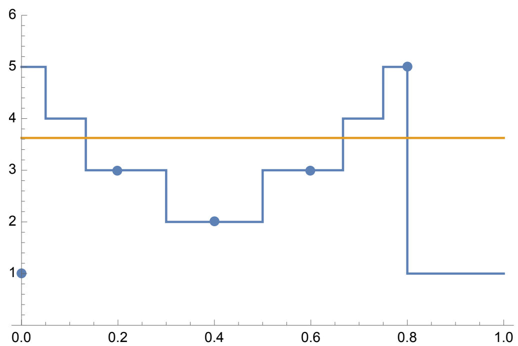


$\delta = 0.2$ $\delta = 0.05$ $\delta = 0.01$

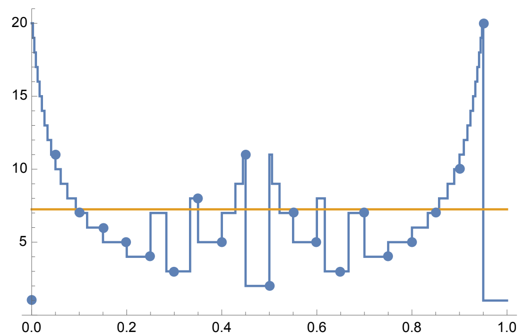
Discrete sampling

$$\tilde{q}_{\text{min}}(x,\delta) = \min\left\{q \in \mathbb{N} : \exists \tfrac{p}{q} \in \mathbb{Q} \cap [x, x+\delta)\right\} \simeq q_{\text{min}}(x + \tfrac{1}{2}\delta, \delta)$$

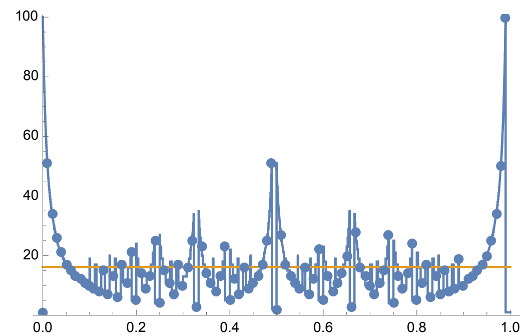
$$\tilde{q}_{\text{min}}(\tfrac{j}{N}, \tfrac{1}{N}) = \min\left\{q \in \mathbb{N} : \exists \tfrac{p}{q} \in \mathbb{Q} \cap [j, j+\tfrac{1}{N})\right\},\ j = 0, \dots, N-1$$



$N = 5$



$N = 20$

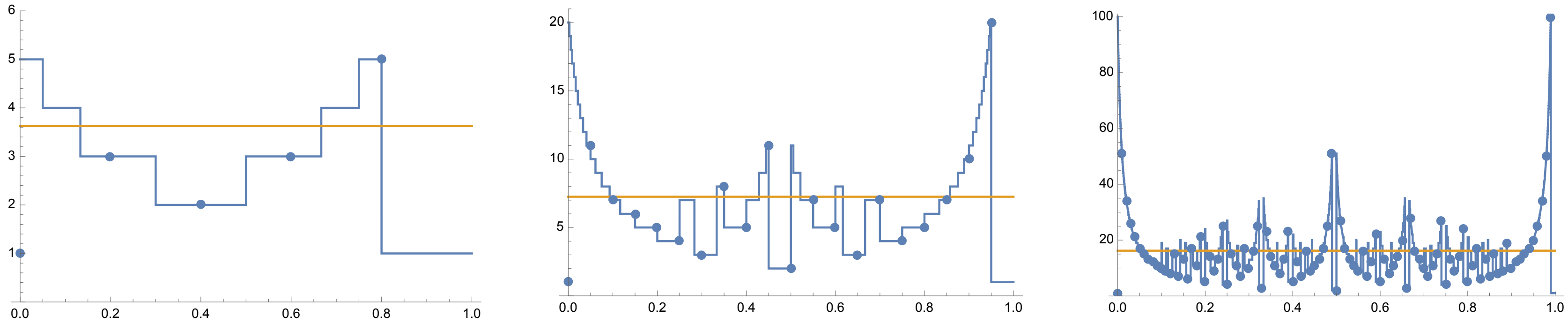


$N = 100$

The Kruyswijk-Meijer conjecture, 1977

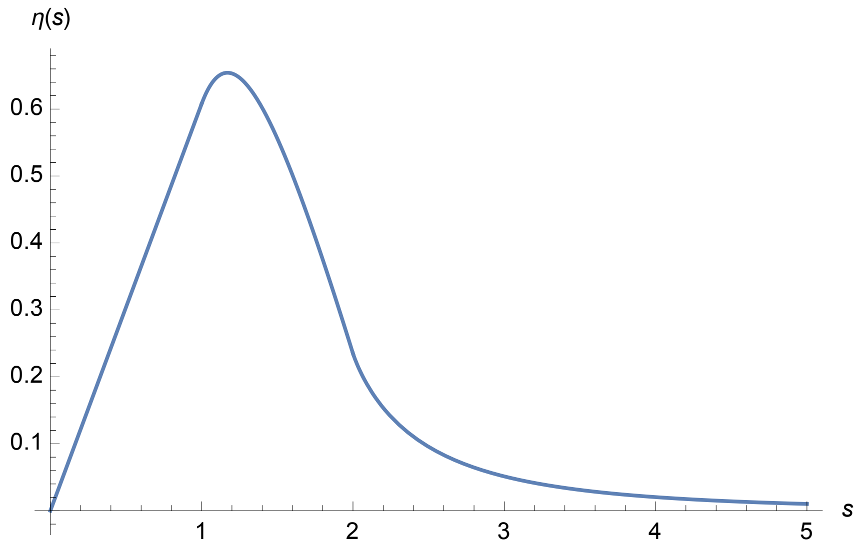
Theorem B. (Balazard & Martin 2023)

$$\frac{1}{N}\sum_{j=0}^{N-1} \tilde{q}_{\min}\left(\tfrac{j}{N}, \tfrac{1}{N}\right) = \frac{16}{\pi^2} N^{1/2} + O(N^{1/3}\log^2 N)$$



$N = 5$ $\quad N = 20$ $\quad N = 100$

The limit distribution (continuous sampling)



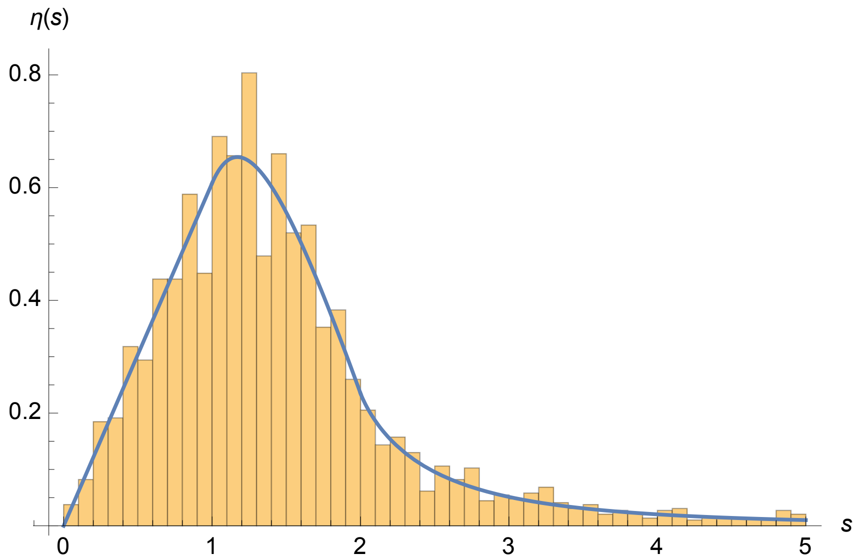
Theorem C. (JM 2024, cf. also Artiles 2023)
For any interval $\mathcal{D} \subset [0,1]$ and $L > 0$, we have

$$\lim_{\delta \to 0} \operatorname{vol} \left\{ x \in \mathcal{D} : \delta^{1/2} q_{\min}(x,\delta) > L \right\} = \operatorname{vol} \mathcal{D} \int_L^\infty \eta(s)\, ds$$

with the probability density

$$\eta(s) = \frac{6}{\pi^2} \times \begin{cases} s & \text{if } s \in [0,1] \\ -s + 2s^{-1} + 4s^{-1} \log s & \text{if } s \in [1,2] \\ -s + 2s^{-1} + 2s\sqrt{\frac{1}{4} - s^{-2}} - 4s^{-1} \log\left(\frac{1}{2} + \sqrt{\frac{1}{4} - s^{-2}}\right) & \text{if } s \geq 2, \end{cases}$$

The limit distribution (continuous sampling)



Data with 3000 random choices of x

Theorem C. (JM 2024, cf. also Artiles 2023)
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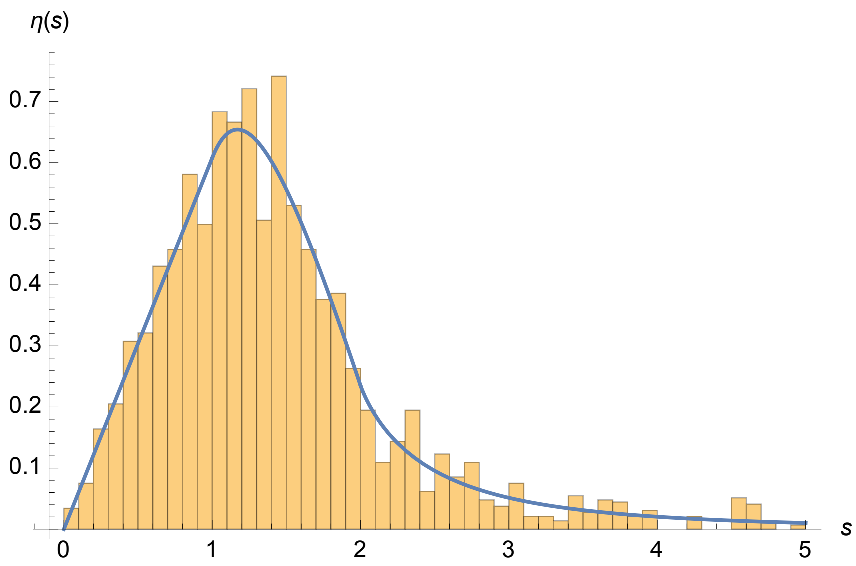
The limit distribution (discrete sampling)



Data with $x = \dfrac{j}{3000}$

$j = 0, 1, 2, \ldots, 2999$

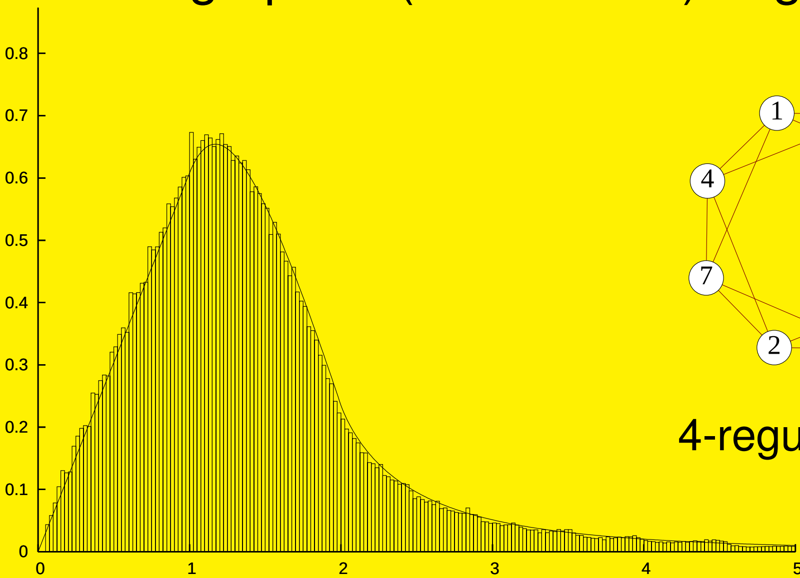
Theorem D. (JM 2024)

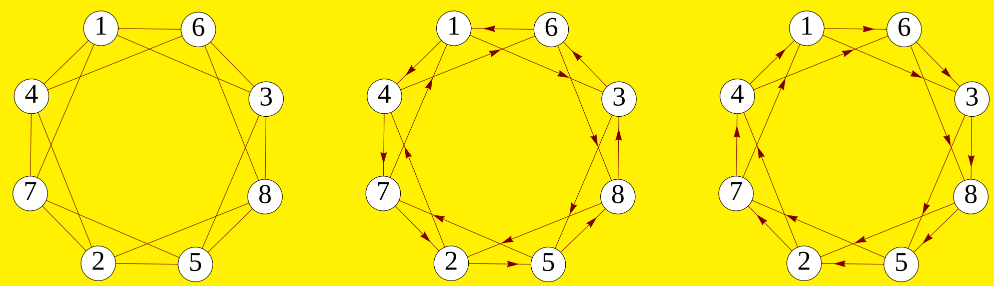
For any interval $\mathcal{D} \subset [0, 1]$ and $L > 0$, we have

$$\lim_{N\to\infty} \frac{\#\left\{\frac{j}{N} \in \mathcal{D} : \tilde{q}_{\min}(\frac{j}{N}, \frac{1}{N}) > LN^{1/2}\right\}}{N \operatorname{vol} \mathcal{D}} = \int_L^\infty \eta(s)\, ds$$

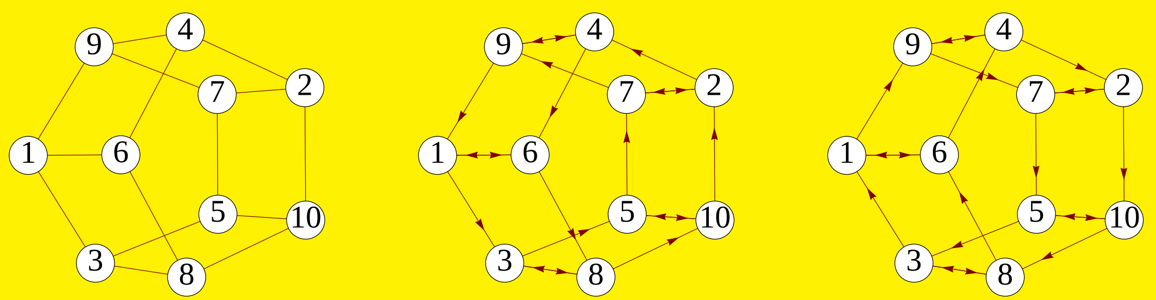
with the same $\eta(s)$ as before.

<u>Note</u>: The same law describes the shortest cycle length of a large random circulant directed graph of (in- and out-) degree 2:





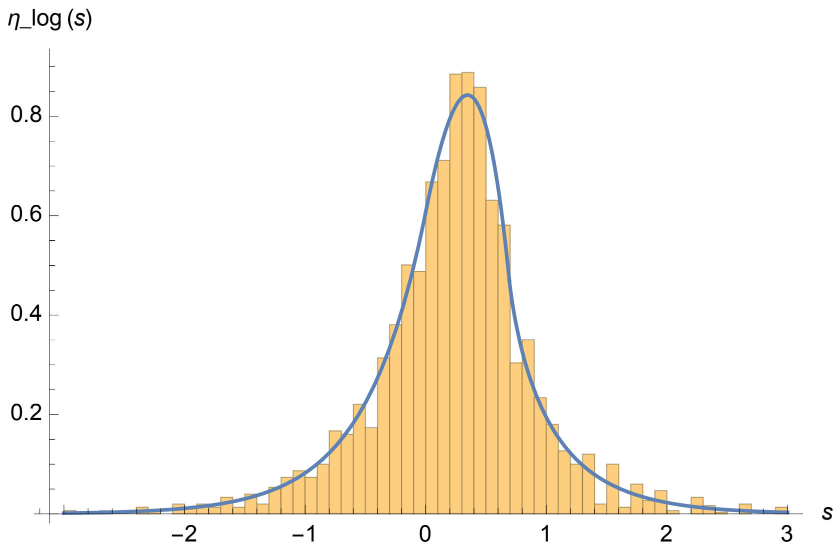
4-regular circulant graph $C_8(2,3)$ and the circulant digraphs $C_8^+(2,3)$, $C_8^+(2,5)$



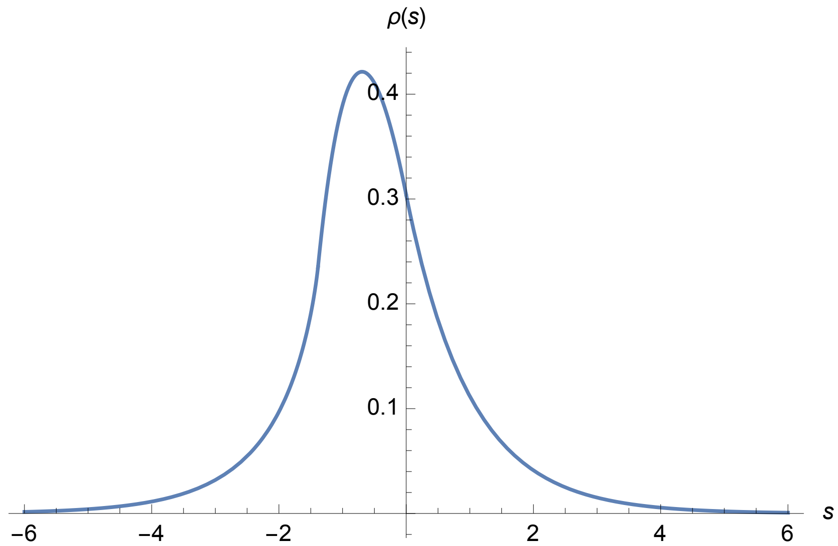
3-regular circulant graph $C_{10}(2,5)$ and the circulant digraphs $C_{10}^+(2,5)$, $C_{10}^+(5,8)$.

JM & Strömbergsson 2013

Extreme events for horocyles



Distribution of log $\tilde{q}_{\min}(\frac{j}{N}, \frac{1}{N})$
$j = 0, \ldots, 2999$



Extreme value law for horocyle flow
$\Gamma = \mathrm{SL}(2, \mathbb{Z})$

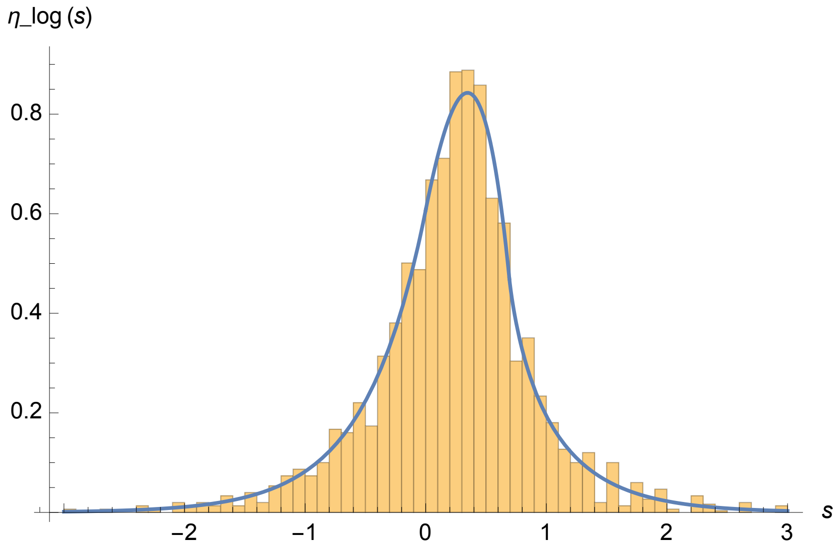
$\mathcal{Y} = \Gamma\backslash\mathbb{H}$ hyperbolic surface with at least one cusp, $\mathcal{X} = T^1(\mathcal{Y})$
h_s horocycle flow on $\mathcal{X}$, μ Liouville measure on $\mathcal{X}$, $\pi : \mathcal{X} \to \mathcal{Y}$ canonical projection

Theorem E. (JM & Pollicott 2024; cf. also Kirsebom & Mallahi-Karai 2022)
Fix $y \in \mathcal{Y}$, Borel probability measure $\lambda \ll \mu$. Then there exists a probability density $\omega_y \in \mathrm{L}^1(\mathbb{R})$ with $\omega_y(s) \asymp \mathrm{e}^{-|s|}$ such that, for every $H \in \mathbb{R}$,

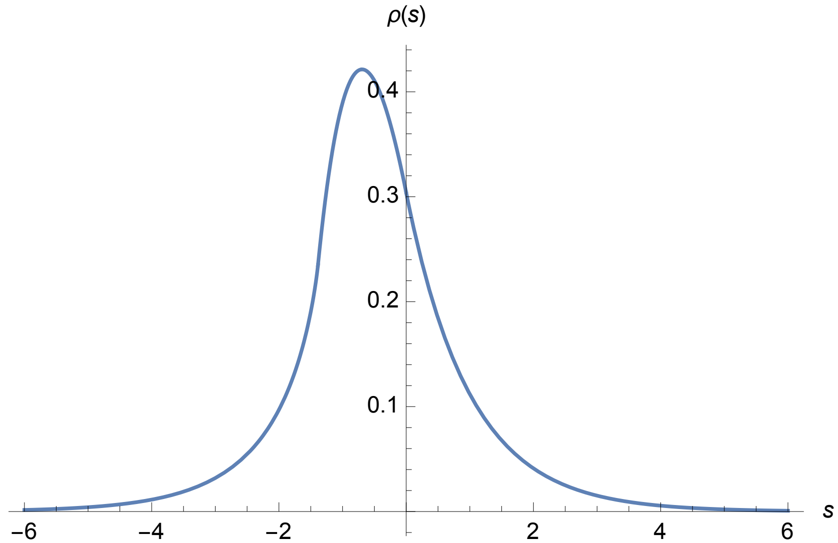
$$\lim_{T\to\infty} \lambda\{x_0 \in \mathcal{X} : \sup_{0<s\leq T} \mathrm{dist}_{\mathcal{Y}}(y, \pi \circ h_s(x_0)) > H + \log T\} = \int_H^\infty \omega_y(s) ds.$$

Dynamical logarithm laws: Sullivan (1982), Kleinbock & Margulis (1999) for geodesics/diagonal actions, ..., Athreya & Margulis (2009), Kelmer & Mohammadi (2012), Yu (2017) for unipotents

Extreme events for horocycles



Distribution of log $\tilde{q}_{\min}(\frac{j}{N}, \frac{1}{N})$
$j = 0, \ldots, 2999$



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h_s horocycle flow on $\mathcal{X}$, μ Liouville measure on $\mathcal{X}$, $\pi : \mathcal{X} \to \mathcal{Y}$ canonical projection

$$\rho(s) = \frac{3}{\pi^2} \times \begin{cases} -e^{-s} + 2 + 2e^{-s}\sqrt{\frac{1}{4} - e^s} - 4\log\left(\frac{1}{2} + \sqrt{\frac{1}{4} - e^s}\right) & \text{if } s \in (-\infty, -2\log 2] \\ -e^{-s} + 2 - 2s & \text{if } s \in [-2\log 2, 0] \\ e^{-s} & \text{if } s \in [0, \infty). \end{cases}$$

Proof of Theorem C

Farey fractions of level Q:

$$\mathcal{F}_Q = \left\{ \frac{p}{q} \in [0,1) : (p,q) \in \widehat{\mathbb{Z}}^2,\ 0 < q \le Q \right\}$$

where $\widehat{\mathbb{Z}}^2$ = set of primitive lattice points. Note: $\#\mathcal{F}_Q \sim \sigma_Q := \frac{3}{\pi^2} Q^2, \quad Q \to \infty$

Key point:

$$\begin{aligned} & q_{\text{min}}(x,\delta) > L\delta^{-1/2} \\ & \Leftrightarrow \left\{ (p,q) \in \widehat{\mathbb{Z}}^2 : 0 < q \le L\delta^{-1/2}, \tfrac{p}{q} \in \left(x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2} \right) \right\} = \emptyset \\ & \Leftrightarrow \mathcal{F}_Q \cap \left(x - \tfrac{s}{2\sigma_Q}, x + \tfrac{s}{2\sigma_Q} \right) + \mathbb{Z} = \emptyset \end{aligned}$$

for $Q = L\delta^{-1/2}$, $s = \frac{3}{\pi^2} L^2$,

As proved by Kargaev & Zhigljavsky (1997) (for $\mathcal{D} = [0,1]$, see JM 2013 for general $\mathcal{D}$), the Lebesgue measure of the set of $x \in \mathcal{D}$ has a limit, namely the void distribution

$$P(s) = \int_s^\infty H(s)\, ds',$$

whose density is the Hall distribution for the gap probabilities in the Farey sequence (Hall 1970). The limit is therefore

$$\int_{\frac{3}{\pi^2}L^2}^\infty H(s)\, ds$$

and the formula for $\eta(s)$ follows by differentiation from Hall's formula. QED

Moments

Theorem F. (JM 2024)
For any interval $\mathcal{D} \subset [0,1]$ and $\alpha \in \mathbb{C}$ with $|\operatorname{Re}\alpha| < 2$, we have

$$\lim_{\delta\to 0} \delta^{\alpha/2} \int_{\mathcal{D}} q_{\min}(x,\delta)^\alpha dx = \operatorname{vol}\mathcal{D}\, M(\alpha) \qquad (*)$$

and

$$\lim_{N\to\infty} N^{-1-\alpha/2} \sum_{j=0}^{N-1} \tilde{q}_{\min}\left(\frac{j}{N}, \frac{1}{N}\right)^\alpha = M(\alpha). \qquad (**)$$

with

$$M(\alpha) = \int_0^\infty s^\alpha \eta(s)\, ds = \frac{24}{\pi^2\alpha(\alpha+2)}\left(\frac{2}{\alpha} + 2^\alpha \mathrm{B}\left(-\frac{\alpha}{2}, \frac{1}{2}\right)\right)$$

The formula for $M(\alpha)$ follows from Kargaev & Zhigljavsky's 1997 study of the void statistics for the Farey sequence. For $\alpha = 1$, $(*)$ implies the Chen-Haynes theorem and $(**)$ the Kruyswijk-Meijer conjecture. Indeed, $M(1) = \frac{16}{\pi^2}$.

Explicit formulas for generalised moments

By dominated convergence, we also get convergence for the generalised moment

$$\lim_{\delta\to 0}\frac{\delta^{\alpha/2}}{\operatorname{vol}\mathcal{D}}\int_{\mathcal{D}} q_{\min}(x,\delta)^{\alpha}\left(\log q_{\min}(x,\delta)+\tfrac{1}{2}\log\delta\right)^{n}dx=\mu_{n,\alpha}$$

for $|\operatorname{Re}\alpha|<2$, $n=0,1,2,\ldots$, and

$$\mu_{n,\alpha}:=\int_0^\infty s^{\alpha}(\log s)^n\eta(s)\,ds=\frac{d^n}{d\alpha^n}\int_0^\infty s^{\alpha}\eta(s)\,ds=\frac{d^nM(\alpha)}{d\alpha^n}$$

so $n=0$ gives moments, $\alpha=0$ logarithmic moments

α	$\mu_{0,\alpha}$	$\mu_{1,\alpha}$	$\mu_{2,\alpha}$	$\mu_{3,\alpha}$
-1	$\frac{12(4-\pi)}{\pi^2}$	$\frac{12(4-\pi\log 4)}{\pi^2}$	$\frac{192-\pi(\pi^2+24+48\log^2 2)}{\pi^2}$	$\frac{6(96-3\pi\zeta(3)-\pi\log 2\,(\pi^2+24+16\log^2 2))}{\pi^2}$
0	1	$\frac{6\zeta(3)}{\pi^2}-\frac{1}{2}$	$\frac{3\pi^2}{40}+\frac{1}{2}-\frac{6\zeta(3)}{\pi^2}$	$\frac{9(\zeta(3)+3\zeta(5))}{\pi^2}-\frac{9\pi^2}{80}-\frac{6\zeta(3)+3}{4}$
1	$\frac{16}{\pi^2}$	$\frac{16(3\pi-7)}{3\pi^2}$	$\frac{32(34+3\pi(6\log 2-7))}{9\pi^2}$	$\frac{4(9\pi^3-1136+48\pi(17+9\log^2 2-21\log 2))}{9\pi^2}$

α	$\mu_{0,\alpha}$	$\mu_{2,\alpha}$
$-\frac{1}{2}$	$\frac{16}{\pi^2}\left(8-\frac{\sqrt{2\pi}\,\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)$	$\frac{16}{9\pi^2}\left(1408-\frac{\sqrt{2\pi}\,\Gamma\left(\frac{5}{4}\right)(144G+9\pi^2+224+12\pi(4+3\log 2)+12\log 2(8+3\log 2))}{\Gamma\left(\frac{3}{4}\right)}\right)$
$\frac{1}{2}$	$\frac{96}{5\pi^2}\left(4+\frac{\sqrt{2\pi}\,\Gamma\left(-\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$	$\frac{24}{125\pi^2}\left(11008+\frac{\sqrt{2\pi}\,\Gamma\left(-\frac{1}{4}\right)(-400G+25\pi^2+440\pi+2752+100\log^2 2-20(44+5\pi)\log 2)}{\Gamma\left(\frac{1}{4}\right)}\right)$

G = Catalan's constant

Coming up. . .

- Smallest denominators in higher dimensions
- Convergence of moments proof of Theorem E (i)
- Discrete sampling proof of Theorem D, Theorem E (ii)
- Moments of the Farey distance function; pigeon hole statistics
- Extreme events for horocycles

Smallest denominators in higher dimensions

- $\mathcal{A} \subset \mathbb{R}^n$ with non-empty interior
- $q_{\min}(\boldsymbol{x}, \delta, \mathcal{A}) = \min \left\{ q \in \mathbb{N} : \exists \frac{\boldsymbol{p}}{q} \in \mathbb{Q}^n \cap \boldsymbol{x} + \delta \mathcal{A} \right\}$
- $G = \mathrm{SL}(n+1, \mathbb{R})$, $\Gamma = \mathrm{SL}(n+1, \mathbb{Z})$
- μ is the Haar probability measure on $\Gamma \backslash G$ and
- $\mathfrak{C}(\mathcal{A}) = \{(\boldsymbol{x}, y) \in \mathbb{R}^n \times (0, 1] : \boldsymbol{x} \in \sigma_1^{-1/n} y \mathcal{A}\} \subset \mathbb{R}^{n+1}$
- $P(0, \mathcal{A}) = \mu\{g \in \Gamma \backslash G : \widehat{\mathbb{Z}}^{n+1} g \cap \mathfrak{C}(\mathcal{A}) = \emptyset\}$
  where $\widehat{\mathbb{Z}}^{n+1}$ set of primitive lattice points in $\mathbb{R}^{n+1}$

Theorem G. (JM 2024, cf. also Artiles 2023)
For $\mathcal{A} \subset \mathbb{R}^n$ bounded and $\mathcal{D} \subset [0, 1]^n$, both with boundary of Lebesgue measure zero and non-empty interior, $L > 0$, we have

$$\lim_{\delta \to 0} \frac{\mathrm{vol} \left\{ \boldsymbol{x} \in \mathcal{D} : \delta^{n/(n+1)} q_{\min}(\boldsymbol{x}, \delta, \mathcal{A}) > L \right\}}{\mathrm{vol}\, \mathcal{D}} = E_{\mathcal{A}}(L)$$

with $E_{\mathcal{A}}(L) = P(0, \sigma_1^{1/n} L^{1+1/n} \mathcal{A})$.

Note $P(0, s\mathcal{A}) \asymp s^{-n}$ as $s \to \infty$ (Strömbergsson 2011), and so $E_{\mathcal{A}}(L) \asymp L^{-(n+1)}$ for $L \to \infty$.

Proof

Farey fractions of level Q:

$$\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^n : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^{n+1},\ 0 < q \leq Q \right\}$$

where $\widehat{\mathbb{Z}}^{n+1}$ = set of primitive lattice points. Note:

$$\#\mathcal{F}_Q \sim \sigma_Q := \frac{Q^{n+1}}{(n+1)\,\zeta(n+1)}, \quad Q \to \infty$$

As in the one-dimensional case, key point is

$$q_{\min}(\boldsymbol{x}, \delta, \mathcal{A}) > L\delta^{-n/(n+1)} \Leftrightarrow \mathcal{F}_Q \cap \boldsymbol{x} + \sigma_Q^{-1/n} s\mathcal{A} + \mathbb{Z}^n = \emptyset, \qquad (\nabla)$$

with $Q = L\delta^{-n/(n+1)}$ and $s = \sigma_1^{1/n} L^{1+1/n}$.

Known results on Farey statistics (JM 2013, based on JM & Strömbergsson 2010) state that the volume of the set of $\boldsymbol{x} \in \mathcal{D}$ satisfying (∇) converges to $P(0, s\mathcal{A})$.

QED

Moments

Theorem H. (JM 2024)
For $\mathcal{A} \subset \mathbb{R}^n$ bounded and $\mathcal{D} \subset [0,1]^n$, both with boundary of Lebesgue measure zero and non-empty interior, $\alpha \in \mathbb{C}$ with $|\operatorname{Re}\alpha| < n+1$, we have

$$\lim_{\delta \to 0} \frac{\delta^{\alpha n/(n+1)}}{\operatorname{vol}\mathcal{D}} \int_{\mathcal{D}} q_{\min}(\boldsymbol{x}, \delta, \mathcal{A})^{\alpha} d\boldsymbol{x} = \int_0^{\infty} L^{\alpha}\, dE_{\mathcal{A}}(L).$$

Proof if $\operatorname{Re}\alpha = 0$

Theorem F can be restated as

$$\lim_{\delta \to 0} \frac{1}{\operatorname{vol}\mathcal{D}} \int_{\mathcal{D}} F(\delta^{n/(n+1)} q_{\min}(\boldsymbol{x}, \delta, \mathcal{A}))d\boldsymbol{x} = \int_0^{\infty} F(L)\, dE_{\mathcal{A}}(L)$$

Apply this with $F(t) = t^{\alpha}$, which is bounded continuous for $\operatorname{Re}\alpha = 0$. QED

Proof if $\operatorname{Re}\alpha > 0$

$$\delta^{\alpha n/(n+1)} \int_{\mathcal{D}} q_{\min}(\boldsymbol{x},\delta,\mathcal{A})^{\alpha} d\boldsymbol{x} = \alpha \int_0^\infty L^{\alpha-1} \operatorname{vol}\left\{\boldsymbol{x}\in\mathcal{D} : \delta^{n/(n+1)} q_{\min}(\boldsymbol{x},\delta,\mathcal{A}) > L\right\} dL$$

Therefore need to show "no escape of mass at infinity":

$$\lim_{R\to\infty} \limsup_{\delta\to 0} \int_R^\infty L^{\operatorname{Re}\alpha-1} \operatorname{vol}\left\{\boldsymbol{x}\in\mathcal{D} : \delta^{n/(n+1)} q_{\min}(\boldsymbol{x},\delta,\mathcal{A}) > L\right\} dL = 0 \qquad (\square)$$

We have

$$\begin{aligned} & q_{\min}(\boldsymbol{x},\delta,\mathcal{A}) > L\delta^{-n/(n+1)} \\ & \Leftrightarrow \mathcal{F}_Q \cap \boldsymbol{x} + \sigma_Q^{-1/n} s\mathcal{A} + \mathbb{Z}^n = \emptyset \\ & \Leftrightarrow \widehat{\mathbb{Z}}^{n+1} h(\boldsymbol{x}) a(Q) \cap \mathfrak{C}(s\mathcal{A}) = \emptyset \\ & \Leftrightarrow \widehat{\mathbb{Z}}^{n+1} h(\boldsymbol{x}) a(\delta^{-n/(n+1)}) \cap L\mathfrak{C}(\sigma_1^{1/n}\mathcal{A}) = \emptyset \end{aligned}$$

with $Q = L\delta^{-n/(n+1)}$, $s = \sigma_1^{1/n} L^{1+1/n}$, and

$$h(\boldsymbol{x}) = \begin{pmatrix} 1_n & {}^{\mathrm{t}}\mathbf{0} \\ -\boldsymbol{x} & 1 \end{pmatrix}, \qquad a(y) = \begin{pmatrix} y^{1/n} 1_n & {}^{\mathrm{t}}\mathbf{0} \\ \mathbf{0} & y^{-1} \end{pmatrix}$$

We thus need to estimate the measure of $\boldsymbol{x}$ for which the primitive lattice $\widehat{\mathbb{Z}}^{n+1} h(\boldsymbol{x}) a(\delta^{-n/(n+1)})$ avoids the cone $L\mathfrak{C}(\sigma_1^{1/n}\mathcal{A})$ for large L. Combining results of Strömbergsson (2011) and Kim & JM (2022), one can show that this is uniformly bounded above by $\ll L^{-(n+1)}$ which in turn implies ($\square$) for $0 < \operatorname{Re}\alpha < n+1$. QED

Proof if $\mathrm{Re}\,\alpha < 0$

$$\delta^{\alpha n/(n+1)} \int_{\mathcal{D}} q_{\min}(\boldsymbol{x},\delta,\mathcal{A})^{\alpha} d\boldsymbol{x} = -\alpha \int_0^{\infty} L^{\alpha-1} \operatorname{vol}\left\{\boldsymbol{x} \in \mathcal{D} : \delta^{n/(n+1)} q_{\min}(\boldsymbol{x},\delta,\mathcal{A}) \leq L\right\} dL.$$

Therefore need to show "no escape of mass at zero":

$$\lim_{r\to 0} \limsup_{\delta\to 0} \int_0^{r} L^{\mathrm{Re}\,\alpha-1} \operatorname{vol}\left\{\boldsymbol{x} \in \mathcal{D} : \delta^{n/(n+1)} q_{\min}(\boldsymbol{x},\delta,\mathcal{A}) \leq L\right\} dL = 0. \qquad (\square)$$

We have

$$\begin{aligned} & q_{\min}(\boldsymbol{x},\delta,\mathcal{A}) \leq L\delta^{-n/(n+1)} \\ & \Leftrightarrow \mathcal{F}_Q \cap \boldsymbol{x} + \sigma_Q^{-1/n} s\mathcal{A} + \mathbb{Z}^n \neq \emptyset \\ & \Leftrightarrow \widehat{\mathbb{Z}}^{n+1} h(\boldsymbol{x}) a(Q) \cap \mathfrak{C}(s\mathcal{A}) \neq \emptyset \\ & \Leftrightarrow \widehat{\mathbb{Z}}^{n+1} h(\boldsymbol{x}) a(\delta^{-n/(n+1)}) \cap L\mathfrak{C}(\sigma_1^{1/n}\mathcal{A}) \neq \emptyset \end{aligned}$$

with $Q = L\delta^{-n/(n+1)}$, $s = \sigma_1^{1/n} L^{1+1/n}$, and

$$h(\boldsymbol{x}) = \begin{pmatrix} 1_n & {}^{\mathrm{t}}\boldsymbol{0} \\ -\boldsymbol{x} & 1 \end{pmatrix}, \qquad a(y) = \begin{pmatrix} y^{1/n} 1_n & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & y^{-1} \end{pmatrix}$$

We thus need to estimate the measure of $\boldsymbol{x}$ for which the primitive lattice $\widehat{\mathbb{Z}}^{n+1} h(\boldsymbol{x}) \delta^{-n/(n+1)}$ has a point in the cone $L\mathfrak{C}(\sigma_1^{1/n}\mathcal{A})$ for small L, so in particular has a short vector of length $\ll L$. A classical estimates yields that this is bounded above by $\ll L^{n+1}$ which turn implies ($\square$) for $-(n+1) < \mathrm{Re}\,\alpha < 0$. QED

Discrete sampling

Recall: $h(\boldsymbol{x}) = \begin{pmatrix} 1_n & {}^t\mathbf{0} \\ -\boldsymbol{x} & 1 \end{pmatrix}$, $\quad a(y) = \begin{pmatrix} y^{1/n} 1_n & {}^t\mathbf{0} \\ \mathbf{0} & y^{-1} \end{pmatrix}$

The following equidistribution theorem is the key input in the continuous sampling case / void statistics of Farey fractions:

Theorem I. (Margulis' thesis 1970, Eskin & McMullen 1993, . . .)

For $f : \Gamma\backslash G \to \mathbb{R}$ bounded continuous, we have

$$\lim_{Q\to\infty} \frac{1}{\operatorname{vol} \mathcal{D}} \int_{\mathcal{D}} f\Big(h(\boldsymbol{x})a(Q)\Big) d\boldsymbol{x} = \int_{\Gamma\backslash G} f(g)\, d\mu(g)$$

In the discrete sampling case we need to replace this by:

Theorem J.

For $f : \Gamma\backslash G \to \mathbb{R}$ bounded continuous, $c > 0$, we have

$$\lim_{\substack{N,Q\to\infty \\ cQ^{n+1}\leq N^n}} \frac{1}{N^n \operatorname{vol} \mathcal{D}} \sum_{\boldsymbol{j}\in\mathbb{Z}^n/N\mathbb{Z}^n \cap N\mathcal{D}} f\Big(h(\boldsymbol{x}_0 + N^{-1}\boldsymbol{j})a(Q)\Big) = \int_{\Gamma\backslash G} f(g)\, d\mu(g)$$

Proof of discrete equidistribution

Define sequence of probability measures ν_i on $\Gamma\backslash G$ by

$$\nu_i(f) = \frac{1}{\#(\mathbb{Z}^n/N_i\mathbb{Z}^n \cap N_i\mathcal{D})} \sum_{\boldsymbol{j}\in\mathbb{Z}^n/N_i\mathbb{Z}^n\cap N_i\mathcal{D}} f\Big(h(\boldsymbol{x}_0 + N_i^{-1}\boldsymbol{j})a(Q_i)\Big)$$

Need to show converges weakly to the probability measure μ.

$$h(\boldsymbol{x}_0 + N^{-1}\boldsymbol{j})a(Q) = h(\boldsymbol{x}_0)a(Q)h(Q^{1+1/n}N^{-1}\boldsymbol{j}) \qquad (\Delta)$$

hence points are finite distance apart (in terms of any left-invariant Riemannian metric on G), and can use escape-of-mass estimate in continuous setting to show $(\nu_i)_i$ is tight and thus each subsequence contains a convergent subsequence. Can now assume w.l.o.g. that f has compact support and is therefore uniformly continuous. Restrict to subsequences along which $Q_i^{1+1/n}N_i^{-1} \to \tau_0$ for some $\tau_0 \in [0, c^{-1/n}]$.

If $\tau_0 = 0$, the discrete average is uniformly close to the continuous average (by uniform continuity of f), and thus the limit is given by μ.

Proof of discrete equidistribution (cont'd)

If $\tau_0 > 0$, then by ($\triangle$) every weak limit is invariant under the map $\Gamma\backslash G \to \Gamma\backslash G$, $\Gamma g \mapsto \Gamma g h(\tau_0 \boldsymbol{j})$ for any $\boldsymbol{j} \in \mathbb{Z}^n$. Since G-action on $\Gamma\backslash G$ by right multiplication is mixing with respect to μ (Howe & Moore 1979), we have that action of the subgroup $H_{\tau_0} = \{h(\tau_0 \boldsymbol{j}) : \boldsymbol{j} \in \mathbb{Z}^n\}$ is μ-ergodic.

Given $\epsilon > 0$, define

$$\nu_i^\epsilon(f) = \nu_i(f_\epsilon), \qquad f_\epsilon(g) := \frac{1}{\epsilon^n} \int_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]^n} f\big(gh(\boldsymbol{x})\big)\, d\boldsymbol{x}.$$

$$\mu_i(f) = \frac{1}{\operatorname{vol} \mathcal{D}} \int_{\mathcal{D}} f\big(h(\boldsymbol{x}) a(Q_i)\big)\, d\boldsymbol{x},$$

$$\overline{\nu}_i^\epsilon = \frac{\mu_i - \epsilon^n \nu_i^\epsilon}{1 - \epsilon^n}.$$

Suppose $\nu_i^\epsilon \to \nu^\epsilon$ along a converging subsequence. As $\mu_i \to \mu$ (along any subsequence), by construction $\overline{\nu}_i^\epsilon \to \overline{\nu}^\epsilon$ along the same subsequence as ν_i^ϵ, and the limits satisfy the relation

$$\epsilon \nu^\epsilon + (1-\epsilon)\overline{\nu}^\epsilon = \mu.$$

All three limit measures are H_{τ_0}-invariant. Since the action of H_{τ_0} is μ-ergodic, by the extremality of ergodic measures, we conclude $\nu^\epsilon = \overline{\nu}^\epsilon = \mu$ for every given $\epsilon > 0$. Because f is uniformly continuous, we have

$$\lim_{\epsilon \to 0} \sup_i |\nu_i(f) - \nu_i^\epsilon(f)| = 0$$

and thus every limit point of $(\nu_i)_i$ must be equal to μ. QED

Moments of the Farey distance function

- $\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^n : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^{n+1},\ 0 < q \leq Q \right\}$
- $\#\mathcal{F}_Q \sim \sigma_Q := \frac{Q^{n+1}}{(n+1)\,\zeta(n+1)}, \quad Q \to \infty$
- $\mathrm{dist}(\boldsymbol{x}, \mathcal{F}_Q) = \min\{\|\boldsymbol{x} + \boldsymbol{r} + \boldsymbol{m}\| : \boldsymbol{r} \in \mathcal{F}_Q,\ \boldsymbol{m} \in \mathbb{Z}^n\}$

Theorem K. (JM 2024)
For $\mathcal{D} \subset [0,1]^n$ with boundary of Lebesgue measure zero and non-empty interior, $\beta \in \mathbb{C}$ with $|\operatorname{Re}\beta| < n$, we have

$$\lim_{Q\to\infty} \frac{\sigma_Q^{\beta/n}}{\mathrm{vol}\,\mathcal{D}} \int_{\mathcal{D}} \mathrm{dist}(\boldsymbol{x}, \mathcal{F}_Q)^{\beta} d\boldsymbol{x} = \int_0^\infty s^\beta \, dF_{\mathcal{B}_1}(s),$$

with $F_{\mathcal{B}_1}(s) = P(0, \mathcal{B}_s) = \mu\{g \in \Gamma\backslash G : \widehat{\mathbb{Z}}^{n+1} g \cap \mathfrak{C}(\mathcal{B}_s) = \emptyset\}$.

This generalizes Kargaev and Zhigljavsky (1997) to higher dimensions.

Pigeon hole statistics for the Farey sequence

- $\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^n : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^{n+1},\ 0 < q \leq Q \right\}$
- $\# \mathcal{F}_Q \sim \sigma_Q := \frac{Q^{n+1}}{(n+1)\,\zeta(n+1)}, \quad Q \to \infty$

Theorem L. (JM 2024)
For $s > 0$, $k \in \mathbb{Z}_{\geq 0}$, we have

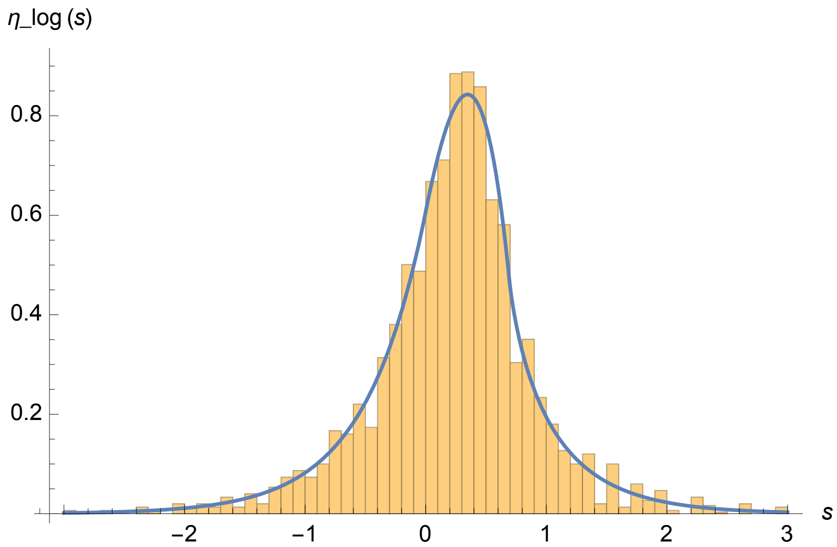
$$\lim_{N \to \infty} \frac{\# \left\{ \boldsymbol{j} \in [0,N)^n : \# \left(\mathcal{F}_Q \cap \frac{\boldsymbol{j}}{N} + \left[0, \frac{1}{N}\right)^n \right) = k \right\}}{N^n} = P(k,s)$$

where $Q = Q_N$ so that $\sigma_Q = s^n N^n$ and

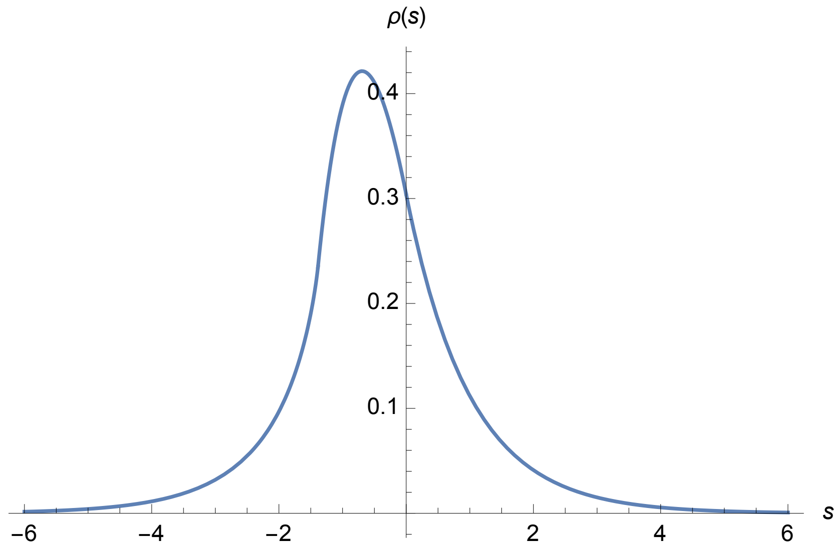
$$P(k,s) = \mu \left\{ g \in \Gamma \backslash G : \#(\widehat{\mathbb{Z}}^{n+1} g \cap \mathfrak{C}([0,s)^n)) = k \right\}.$$

See also Pattison (2023) for pigeonhole statistics of $\sqrt{n} \bmod 1$.

Proof of extreme value theorem for horocyles



Distribution of log $\tilde{q}_{\min}(\frac{j}{N}, \frac{1}{N})$
$j = 0, \ldots, 2999$



Extreme value law for horocycle flow
$\Gamma = \mathrm{SL}(2, \mathbb{Z})$

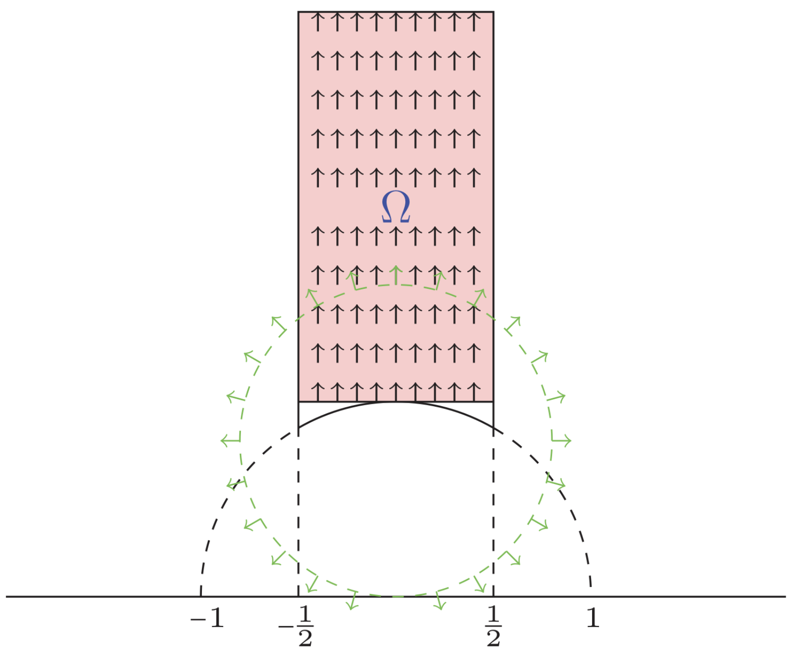
$\mathcal{Y} = \Gamma\backslash\mathbb{H}$ hyperbolic surface with at least one cusp, $\mathcal{X} = T^1(\mathcal{Y})$
h_s horocycle flow on $\mathcal{X}$, μ Liouville measure on $\mathcal{X}$, $\pi : \mathcal{X} \to \mathcal{Y}$ canonical projection

Theorem E. (JM & Pollicott 2024; cf. also Kirsebom & Mallahi-Karai 2022)
Fix $y \in \mathcal{Y}$, Borel probability measure* $\lambda \ll \mu$. Then there exists a probability density $\omega_y \in \mathrm{L}^1(\mathbb{R})$ with $\omega_y(s) \asymp \mathrm{e}^{-|s|}$ such that, for every $H \in \mathbb{R}$,

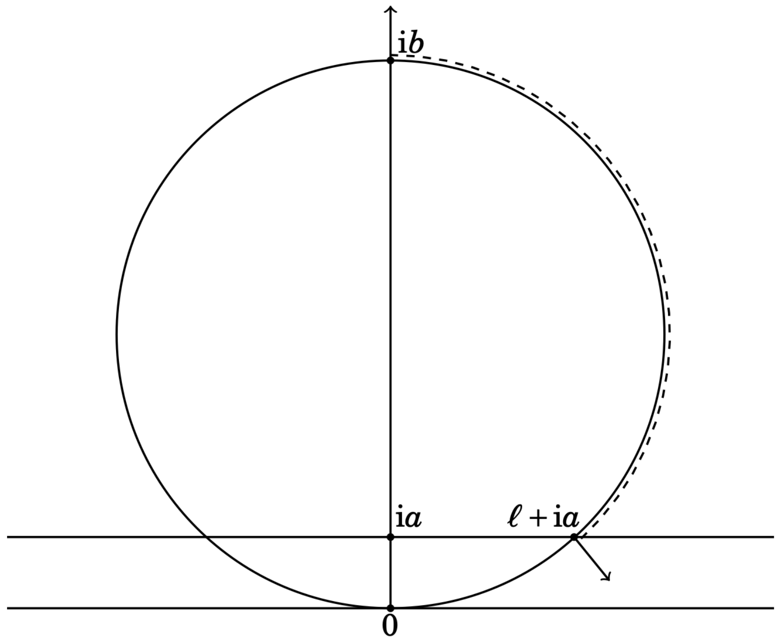
$$\lim_{T\to\infty} \lambda\{x_0 \in \mathcal{X} : \sup_{0<s\leq T} \mathrm{dist}_{\mathcal{Y}}(y, \pi \circ h_s(x_0)) > H + \log T\} = \int_H^\infty \omega_y(s) ds.$$

*can allow for a general class of more singular measures

Proof of extreme value theorem for horocyles



(from Athreya & Cheung 2014)



1. Key idea: relate extreme events to hitting times of the horocycle flow to Athreya & Cheung's Poincaré section truncated high in the cusp
2. Show that the the distance from entering the cusp to hitting the section is relatively small
3. Use the scaling property of the section under the geodesic flow to pull it back from the cusp
4. Mixing of geodesic flow implies that the pushforward of λ under geodesic flow converges to Haar probability measure μ
5. Why is the extreme value law the same as the log distribution of smallest denominators? The return times for the horocycle flow with μ random initial data give the limit distribution of the Farey sequence – which in turn gives the limit distribution of small denominators

Extreme events for horospherical actions

- $G = \mathsf{SL}(n,\mathbb{R})$, $\Gamma = \mathsf{SL}(n,\mathbb{Z})$, $\mathcal{X} = \Gamma\backslash G$

- $\mathbb{R}^k$ action $h_s(x) = xU(s), \quad U(s) = \begin{pmatrix} 1_m & 0 & 0 \\ 0 & 1_k & 0 \\ 0 & -s & 1 \end{pmatrix}, \quad s \in \mathbb{R}^k$

- "distance" from the "origin" $o = \Gamma \simeq \mathbb{Z}^n$ to $x = \Gamma g \simeq \mathbb{Z}^n g$:

$$\alpha_1(x) = \max_{v \in \mathbb{Z}^n g \setminus \{0\}} \frac{1}{\|v\|}$$

- Athreya & Margulis (2009, 2017):

$$\limsup_{T\to\infty} \sup_{\|s\|<T} \frac{\log \alpha(h_s(x_0))}{\log T} = \frac{k}{n}.$$

Extreme events for horospherical actions

Theorem M. (JM 2025)

For Borel probability measure $\lambda \ll \mu$

$$\lim_{T\to\infty} \lambda\left\{x \in \mathcal{X} : \sup_{\|s\|<T} \log \alpha(h_s(x)) > Y + \frac{k}{n} \log T\right\} = D_k(Y),$$

where complementary distribution function $D_k : \mathbb{R} \to [0,1]$ is continuous, and there exist constants $0 < C_1 < C_2 < \infty$ such that

$$C_1 \mathrm{e}^{-nY} \leq D_k(Y) \leq C_2 e^{-nY}, \qquad Y \geq 0,$$

$$C_1 \mathrm{e}^{-n|Y|} \leq 1 - D_k(Y) \leq C_2 \mathrm{e}^{-n|Y|}, \qquad Y \leq 0.$$

Same proof strategy as for $n = 2$. Continuity and tail estimates follow (respectively) from JM-Strömbergsson (2010), Strömbergsson (2011), JM-Strömbergsson (2014) via the formula

$$D_k(Y) = \mu\left(\{\Gamma g \in \Gamma\backslash G : \mathbb{Z}^n g \cap \mathcal{C}(\mathrm{e}^{-Y}) \neq \emptyset\}\right)$$

where $\mathcal{C}(\sigma)$ is a certain cone of volume σ^n.

Related to void distribution of directions in lattices and multi-dimensional Farey fractions.

Further reading

- A. Artiles, The minimal denominator function and geometric generalizations, arXiv:2308.08076
- J.S. Athreya and Y. Cheung, A Poincaré section for horocycle flow on the space of lattices, Int. Math. Res. Not. IMRN (2014), no.10, 2643–2690
- M. Kirsebom and K. Mallahi-Karai, On the extreme value law for the unipotent flow on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, arXiv:2209.07283
- J. Marklof, Smallest denominators, Bull. Lond. Math. Soc. 56 (2024), 1920–1938
- J. Marklof and M Pollicott, Extreme events for horocycle flows, arXiv:2408.01781
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- S. Pattison, Rational points on nonlinear horocycles and pigeonhole statistics for the fractional parts of $\sqrt{n}$, Ergodic Theory Dynam. Systems 43 (2023), no. 9, 3108–3130