Geometric estimates along the Kähler-Ricci flow

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Introductory workshop

The Ricci flow

The Ricci flow, first introduced by Hamilton, is the evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

evolving a Riemannian metric by its Ricci curvature.

- Success : use RF to classify closed 3-dim riem. manifolds (Perelman).
- Obs : Kähler property preserved (Bando) $\to$ KRF.
- Hope : use the KRF to classify compact Kähler manifolds.

Problem : requires efficient tools for weak KRF on singular varieties.

- Lecture 1 : recent geometric estimates for smooth KRF.
- Lecture 2 : properties of Kähler Green's functions.

Kähler cone

Let X be a compact complex manifold of complex dimension n.

- $\omega =_{loc} \sum_{\alpha,\beta} g_{\alpha\overline{\beta}} i dz_\alpha \wedge d\overline{z}_\beta$ is Kähler if $(g_{\alpha\overline{\beta}}) > 0$ and $d\omega = 0$.
- Induces a deRham cohomology class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$.
- Kähler cone $\mathcal{K}$ is open and convex in $H^{1,1}(X, \mathbb{R})$.

Example

If $X = \mathbb{P}^1 \times \mathbb{P}^1$ we set $\alpha_1 = \pi_1^* \alpha_{FS}$ and $\alpha_2 = \pi_2^* \alpha_{FS}$. Then $\mathcal{K} = \{a_1\alpha_1 + a_2\alpha_2 \; a_1, a_2 > 0\} \subset H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{R}) = \mathbb{R}^2$.

Example

If $\pi : X \to \mathbb{P}^2$ is the blow up at a point $p \in \mathbb{P}^2$, set $\alpha_1 = \pi^* \alpha_{FS}$ and $\alpha_2 = \pi^* \alpha_{FS} - \{E\}$. Then $\mathcal{K} = \{a_1\alpha_1 + a_2\alpha_2 \; a_1, a_2 > 0\} \subset H^{1,1}(X, \mathbb{R})$.

Canonical bundle

- $\mathrm{Ric}(\omega) =_{loc} -\frac{1}{\pi}\sum \frac{\partial^2 \log\det(g_{p\bar q})}{\partial z_\alpha \partial \bar z_\beta} i dz_\alpha \wedge d\bar z_\beta$ =Ricci curvature;
- $\mathrm{Ric}(\omega)$ =globally well defined closed $(1,1)$-form representing $c_1(X)$.

Definition

- *The canonical bundle K_X is the line bundle of holomorphic n-forms.*
- *Its first Chern class is $c_1(K_X) = -\{\mathrm{Ric}(\omega)\} = -c_1(X)$.*

Example

Canonical bdle of smooth hypersurface of degree d in $\mathbb{P}^n$ is $\mathcal{O}(d-n-1)$.
Trichotomy: either $d < n+1$, or $d = n+1$, orelse $d > n+1$.

Definition

A cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$ is nef if it is a limit of Kähler classes.
An algebraic variety X is called a minimal model if $c_1(K_X)$ is nef.

Positivity properties of K_X govern the classification

Example

If $n = 2$ and $\pi : X \to Y$ is the blow up at a point $p \in Y$, then $K_X = \pi^* K_Y + E$. In particular $K_X \cdot E = -1$ hence K_X is not nef.

Conjecture (after birational surgeries)

- *either K_X is nef and $\exists f : X \to Y$ with $c_1(X_y) = 0$ and $c_1(K_Y) > 0$;*
- *or $\exists f : X \to Y$ with Fano fibers $c_1(K_{X_y}) < 0$.*

- birational surgeries=blowing down curves in dimension 2.
- OK in dim ≤ 3 [Mori 88] and [Höring-Peternell 16].
- Smooth minimal models do not always exist in dimension $n \geq 3$.
- Abundance conjecture: K_V nef and l.t. sing. $\Rightarrow K_V$ semi-ample ?
- Building blocks: either $K_X > 0$, or $K_X \sim 0$, orelse $K_X < 0$.

Kähler-Ricci flow

Fix ω_0 a Kähler form and consider the Kähler-Ricci flow

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) \quad \text{with} \quad \omega_{|\mathrm{t}=0} = \omega_0$$

- [Hamilton 82, DeTurck 83] : short time existence of the flow.
- Cohom level $\dot{\alpha}_t = c_1(K_X)$ and $\alpha_0 = \{\omega_0\}$, so $\alpha_t = \alpha_0 + tc_1(K_X)$.

Definition

We set $T_{max} = \sup\{t > 0\,;\, \alpha_t = \alpha_0 + tc_1(K_X)$ *is a Kähler class*$\}$.

- If $T_{max} = +\infty$ then $c_1(K_X) = \lim_{t \to +\infty} \frac{\alpha_t}{t}$ is nef.
- Conversely $c_1(K_X)$ nef $\Rightarrow c_1(K_X) > -\varepsilon\alpha_0$ for all $\varepsilon > 0$ hence $\alpha_0 + tc_1(K_X) > (1 - \varepsilon t)\alpha_0 > 0$ if $t < \frac{1}{\varepsilon}$, thus $T_{max} = +\infty$.

Maximal existence

Theorem (Cao 85, Tsuji 88, Tian-Zhang 06)

The Kähler-Ricci flow admits a unique solution $\omega = \omega(t,x) = \omega_t(x)$ *on a maximal domain* $[0, T_{max}[\times X$, *where*

$$T_{max} = \sup\{t > 0\,;\, \{\omega_0\} - tc_1(X) \text{ is Kähler }\}.$$

Moreover $T_{max} = +\infty$ *iff* K_X *is nef (smooth minimal model).*

- Much simpler than in the Riemannian case.
- T_{max} only depends on positivity properties of $c_1(K_X)$ and α_0.
- Can start the KRF from a singular initial datum ω_0 [Di Nezza-Lu 17].
- Same result on sing. varieties [Song-Tian 17, Eyssidieux-G-Zeriahi 16].
- Note : if X is a Fano manifold (i.e. $c_1(X) > 0$) then $T_{max} < +\infty$.

Reduction to a parabolic Monge-Ampère equation

Since $\alpha_t = \alpha_0 - tc_1(X) = \{\omega_0\} - t\{\mathrm{Ric}(\omega_0)\}$, seek for

$$\omega_t = \omega_0 - t\mathrm{Ric}(\omega_0) + \mathrm{dd^c}\varphi_\mathrm{t},$$

where $d = \partial + \overline{\partial}$ and $d^c = \frac{(\partial - \overline{\partial})}{2i\pi}$ hence $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. We infer

$$-\mathrm{Ric}(\omega_0) + \mathrm{dd^c}\dot{\varphi}_\mathrm{t} = \frac{\partial \omega_\mathrm{t}}{\partial \mathrm{t}}$$

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$$-\mathrm{Ric}(\omega_0) + dd^c\dot{\varphi}_t = \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) = -\mathrm{Ric}(\omega_0) + dd^c \log \frac{\omega_t^n}{\omega_0^n}.$$

Thus KRF equivalent to scalar *parabolic complex Monge-Ampère equation*

$$(\omega_0 - t\mathrm{Ric}(\omega_0) + dd^c\varphi_t)^n = e^{\dot{\varphi}_t}\omega_0^n, \quad \text{with} \quad \varphi_0 \equiv 0.$$

Smooth solutions of scalar parabolic equations

More generally fix $0 < T < +\infty$ and

- let $(\theta_t)_{0 \leq t < T}$ be a smooth family of closed differential forms representing a smooth path of Kähler classes $\alpha_t \in \mathcal{K} \subset H^{1,1}(X, \mathbb{R})$;
- let $h(t, x)$ be a smooth function and dV_X a volume form;
- fix $\psi_0 \in \mathcal{K}_{\theta_0}$ an arbitray Kähler potential, i.e. $\theta_0 + dd^c\psi_0 \in \mathcal{K}$.

Theorem

There exists, for all $0 \leq t < T$, a unique $\varphi_t \in \mathcal{K}_{\theta_t}$ such that

$$(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t + h(t,x)} dV_X \quad \textit{with} \quad \varphi_0 \equiv \psi_0.$$

- One can also consider smooth functions $h(t, x, r)$ and equations

$$(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t + h(t,x,\varphi_t)} dV_X.$$

Sketch of proof

Continuity method and a priori estimates:

- There are three global estimates: fix $T' < T$,
  1. there exists $C_0 > 0$ s.t. $|\varphi(t,x)| \leq C_0$ for all $(t,x) \in [0, T'] \times X$.
  2. there exists $C_1 > 0$ s.t. $|\dot{\varphi}(t,x)| \leq C_1$ for all $(t,x) \in [0, T'] \times X$.
  3. there exists $C_2 > 0$ s.t. $\Delta_{\omega_0}\varphi(t,x) \leq C_2$ for all $(t,x) \in [0, T'] \times X$.
- Then use (local) complex parabolic Evans-Krylov theory to obtain

$$||\varphi||_{\mathcal{C}^{2,\frac{\alpha}{2},\alpha}}([0, T'] \times X) \leq C_\alpha$$

  where $0 < \alpha < 1$ and $C_\alpha > 0$.
- Conclude by parabolic Schauder estimates and bootstrapping.

Calabi-Yau manifolds

- Assume $K_X = 0$, so cohomology class $\alpha_t = \alpha_0$ is constant.
- In particular $T_{max} = +\infty$, hence X is a smooth minimal model.
- There exists a unique Kähler Ricci flat metric $\omega_{KE} \in \alpha_0$ [Yau 78].

Theorem (Cao 85)

The Kähler-Ricci flow $\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$ *with arbitrary initial datum* ω_0 *exists for all* $t > 0$ *and smoothly cv to the Calabi-Yau metric* $\omega_{KE} \in \{\omega_0\}$.

- Yields a parabolic proof of Yau's solution to the Calabi conjecture.
- Asymptotic behavior only depends on initial cohomology class.
- Can be extended to non smooth minimal models [ST 17, EGZ 16].

Ample canonical bundle

- Assume $K_X > 0$, hence $T_{max} = +\infty$ and X smooth minimal model.
- $\exists!$ K-E metric $\omega_{KE} \in c_1(K_X)$ [Aubin/Yau 78], $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$.

Theorem (Cao 85)

Fix ω_0 an arbitrary Kähler form on X. The Normalized Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t$$

with initial datum ω_0 exists for all $t > 0$ and smoothly converges to ω_{KE}.

- From KRF to NKRF rescaling $\omega_t = \lambda(t)\tilde{\omega}_{s(t)}$ with $\lambda' = -\lambda$, $\lambda s' = 1$.
- NKRF $\alpha_t = e^{-t}\alpha_0 + (1 - e^{-t})c_1(K_X) \Longrightarrow$ normalized volumes.
- Can be extended to non smooth minimal models [ST 17, EGZ 16].

Fano manifolds

- Assume $K_X < 0$ so $T_{max} < +\infty$, and fix ω_0 a Kähler form in $c_1(X)$.

Theorem (Perelman 03)

If $\exists$ a unique Kähler-Einstein metric then the Normalized Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t, \quad \textit{with initial datum } \omega_0 \in c_1(X)$$

exists for all $t > 0$ and smoothly cv to the K-E metric $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$.

- Similar result by [Tian-Zhu 07] when there exists a K-R-Soliton.
- Can be extended to $\mathbb{Q}$-Fano varieties [Boucksom-Berman-EGZ 19].
- Hamilton-Tian conjecture: $(X, d_{\omega_{t_j}})$ G-H cv to a KRS on (X_∞, d_∞).
  $\hookrightarrow$ proved by [Tian-Zhang 16], [Bamler 18] and [Chen-Wang 20].

An ambitious program

Difficult pbm : understand asymptotic behavior of (X, ω_t) as $t \to T_{max}$.
[Song-Tian 17] have proposed the following conjectural scenario :

- If $T_{max} < +\infty$ show that (X, ω_t) converges to a mildly singular Kähler variety (X_1, S_1) equipped with a singular *Kähler metric* S_1.
- Try and restart the KRF on X_1 with initial data S_1.
- Repeat ftly many times to reach either dim $< n$ or minimal model X_r.
- If dim $< n$ proceed by induction on dimension.
- If K_{X_r} is nef study the long term behavior of the NKRF,

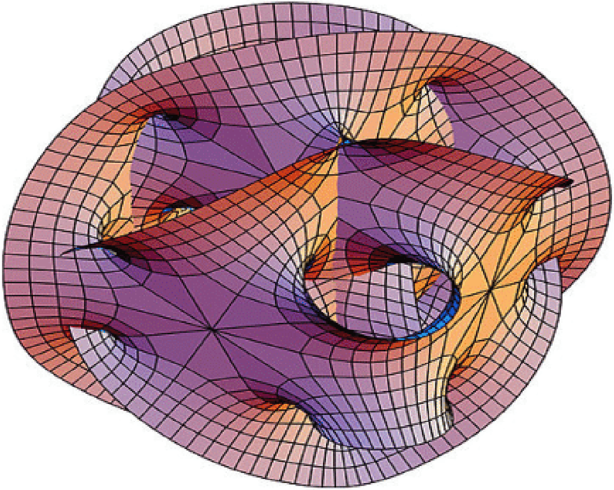
$$\begin{cases} \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega_{\mathrm{t}} \\ \omega_{|t=0} = S_r \end{cases}$$

and show that (X_r, ω_t) converges to a canonical model (X_{can}, ω_{can}).

Known results

- Program achieved in dimension one [Hamilton 86, Chow 91].
- More or less complete in dimension two (...[Song-Weinkove 13]).
- Program largely open in dimension ≥ 3.
- Many difficulties to overcome, among them
  - Degenerate initial data (Kähler current rather than Kähler form).
    OK by [...Di Nezza-Lu 17].
  - Define the KRF on mildly singular varieties.
    OK by [Song-Tian 17...G-Lu-Zeriahi 20].
  - Construct canonical limits and prove convergence.
    In progress [Song-Tian 12, Tosatti-Weinkove-Yang 18,...].
- Focus today: geometric cv of (X, d_{ω_t}) [Guo-Phong-Song-Sturm 24].

Short break: a 3d Calabi-Yau manifold



Diameter bounds and non collapsing

Lecture 1' goal=proof of geometric estimates along the smooth NKRF:

Theorem (Guo-Phong-Song-Sturm 24)

Fix $\varepsilon > 0$. There is $c_\varepsilon, D_0 > 0$ s.t. for all $0 \leq t < T_{max}, p \in X, 0 < r < 1$,

- $\mathrm{diam}(X, \omega_t) \leq D_0$ *and*
- $c_\varepsilon r^{2n+\varepsilon} \mathrm{Vol}_{\omega_t}(X) \leq \mathrm{Vol}_{\omega_t}(B_{\omega_t}(p, r))$

along the NKRF if

- *either $T_{max} = +\infty$ (smooth minimal model, global collapsing OK),*
- *or $T_{max} < +\infty$ and $\mathrm{Vol}_{\omega_{T_{max}}}(X) > 0$ (global non-collapsing).*

- Estimating $\mathrm{Vol}_{\omega_t}(X)$ is easy=cohomological computation (next slide).
- Open problem: diameter bound for collapsing finite time singularity.
  $\hookrightarrow$ Perelman's diameter bound for Fano manifolds=particular case.

Computing global volumes

- NKRF $\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t$ hence $\alpha_t = e^{-t}\alpha_0 + (1-e^{-t})c_1(K_X)$.
- If $T_{max} = +\infty$ then $\mathrm{Vol}_{\omega_t}(X) \sim e^{-(n-\nu(X))t}$ where
  $\nu(X) =$ numerical dim. of $K_X = \sup\{k \in \mathbb{N},\ c_1(K_X)^k \neq 0\}$.
- If $T_{\max} < +\infty$ then
  - either $\mathrm{Vol}_{\omega_t}(X) \sim 1$ if $\alpha^n_{T_{max}} > 0$ (birational surgery),
  - or $\mathrm{Vol}_{\omega_t}(X) \longrightarrow 0$ if $\alpha^n_{T_{max}} = 0$ (collapsing to lower dimension).

Example

If $X = S_1 \times S_2$ with S_j=Riemann surface of genus $g_1, g_2 \geq 1$. Then $n = 2$, $K_X = \pi_1^* K_{S_1} \otimes \pi_2^* K_{S_2} \geq 0$, hence $T_{max} = +\infty$ (minimal model) and

- *$\nu(X) = 0$ if $g_1 = g_2 = 1$; $\nu(X) = 1$ if $g_1 = 1$ and $g_2 \geq 2$;*
- *$\nu(X) = 2$ if $g_1 = g_2 \geq 2$ (X is "of general type" if $\nu(X) = n$).*

Kähler Green's functions

- Key tool=properties of Green's functions associated to Kähler forms.

Definition

Given ω Kähler form we consider $G^\omega \in \mathcal{C}^\infty(X \times X \setminus \mathrm{Diag}, \mathbb{R})$ s.t.

- $G^\omega(x,y) = G^\omega(y,x)$ *for all* $(x,y) \in X \times Y$;
- $G^\omega(x,y) \sim -\frac{1}{[d_\omega(x,y)]^{2n-2}}$ *if* $n \geq 2$;
- *the functions* $y \mapsto G^\omega_x(y) = G^\omega(x,y)$ *are ω-subharmonic with*

$$\frac{1}{V_\omega}(\omega + dd^c G^\omega_x) \wedge \omega^{n-1} = \delta_x,$$

where $V_\omega = \int_X \omega^n$ *and* δ_x*=Dirac mass at point* x.

- Key result: uniform integral bds on ∇G^ω under L^p bds on $f_\omega = \frac{\omega^n / V_\omega}{\omega_X^n}$.

Estimates for Green's functions

Theorem (Guo-Phong-Song-Sturm 24 / G-Tô 24 / Vu 24)

Let (X,ω_X) be a compact Kähler manifold of cplx dim n with $\int_X \omega_X^n = 1$. Fix $A, B > 0$ and $p > 1$. Let ω be another Kähler form such that

$$\int_X \omega \wedge \omega_X^{n-1} \leq A \quad \text{and} \quad \int_X f_\omega^p \omega_X^n \leq B, \quad \text{where} \quad f_\omega = V_\omega^{-1}\omega^n/\omega_X^n.$$

Fix $r < \frac{n}{n-1}$, $s < \frac{2n}{2n-1}$. There exists $C, D > 0$ such that for all $x \in X$,

$$\int_X |G_x^\omega|^r \frac{\omega^n}{V_\omega} \leq C(p,r,A,B) \quad \text{and} \quad \int_X |\nabla G_x^\omega|^s \frac{\omega^n}{V_\omega} \leq D(p,s,A,B).$$

- $\int_X \omega \wedge \omega_X^{n-1} \leq A \Longrightarrow \mathrm{Vol}_\omega(X) \leq V(A)$.
- $\int_X |\nabla G_x^\omega|^s \frac{\omega^n}{V_\omega} \leq D_s \Longrightarrow$ bds on diameter+local non collapsing.
- Proof of these results=Lecture 2.

Parabolic Monge-Ampère equation

NKRF can be reduced to scalar parabolic complex Monge-Ampère equation

$$\frac{1}{V_t}\omega_t^n = \frac{1}{V_t}(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t + \varphi_t + h(x)}\omega_X^n$$

where

- $\theta_t = e^{-t}\omega_0 + (1 - e^{-t})\eta$, with $\eta \in c_1(K_X)$;
- h is a smooth (fixed) function, $V_t = \int_X \omega_t^n$;
- $t \mapsto \varphi_t$ is the unknown function (Kähler potential); $\varphi_0 \equiv 0$.

PLAN:

- observe that $\alpha_t = \{\omega_t\} = \{\theta_t\}$ remains bounded in $H^{1,1}(X, \mathbb{R})$;
- show that $\dot{\varphi}_t(x) + \varphi_t(x) \leq C$ and apply previous thm ($p = +\infty$).

Upper bound on φ_t

Lemma 1

There exists $C_0 > 0$ such that $\varphi_t(x) \leq C_0$ for all $(t, x) \in [0, T_{max}[\times X$.

- If $I(t) = \int_X \varphi_t \omega_X^n$ then $I(t) \leq \sup_X \varphi_t \leq I(t) + C_0'$ (qpsh functions).
- By concavity of the log (Jensen's inequality), we obtain

$$\begin{aligned} I'(t) + I(t) + \int_X h\omega_X^n &= \int_X (\dot{\varphi}_t + \varphi_t + h(x))\omega_X^n \\ &= \int_X \log\left(\frac{\omega_t^n/V_t}{\omega_X^n}\right)\omega_X^n \\ &\leq \log \int_X \left(\frac{\omega_t^n/V_t}{\omega_X^n}\right)\omega_X^n \end{aligned}$$

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$$
\begin{aligned}
I'(t) + I(t) + \int_X h\omega_X^n &= \int_X (\dot{\varphi}_t + \varphi_t + h(x))\omega_X^n \\
&= \int_X \log\left(\frac{\omega_t^n / V_t}{\omega_X^n}\right)\omega_X^n \\
&\le \log \int_X \left(\frac{\omega_t^n / V_t}{\omega_X^n}\right)\omega_X^n = 0.
\end{aligned}
$$

- Since $I(0) = 0$ we infer $I(t) \le C_0'' = -\int_X h\omega_X^n$. $\square$

Upper bound on $\dot\varphi_t$

Lemma 2

There exists $C_1 > 0$ such that $\dot\varphi_t(x) \le C_1$ for all $(t,x) \in [0, T_{max}[\times X$ if

- *either $T_{max} = +\infty$,*
- *or $T_{max} < +\infty$ and $V_{T_{max}} > 0$.*

- Consider $H(t,x) = (e^t - 1)\dot\varphi_t(x) - \varphi_t(x) - b(t)$.
- We choose b s.t. $b(0) = 0$ and $b'(t) = n - (e^t - 1)\frac{d}{dt} \log V_t$.
- Exercise: $T_{max} = +\infty$ (or $T_{max} < +\infty, V_{T_{max}} > 0) \Longrightarrow b(t) \le C_1' e^t$.

Set $\Delta_t u := \Delta_{\omega_t} u = n\frac{dd^c u \wedge \omega_t^{n-1}}{\omega_t^n}$ and $\mathrm{Tr}_{\omega_t}\theta := n\frac{\theta \wedge \omega_t^{n-1}}{\omega_t^n}$.

- Lemma 3: $\left(\frac{\partial}{\partial t} - \Delta_t\right) H \le 0$. Max pple $\Rightarrow H \le \sup_{x\in X} H(0,x) = 0$.
- Lemma 1 + Exercise + Lemma 3 yield $(e^t - 1)\dot\varphi_t(x) \le C_1'' e^t$. □

Upper bound on $\dot{\varphi}_t$ (end)

Lemma 3

If $H = (e^t - 1)\dot{\varphi}_t - \varphi_t - b(t)$ *and* $b'(t) = n - (e^t - 1)\frac{d}{dt}\log V_t$*, then*

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H = -\mathrm{Tr}_{\omega_t}(\omega_0) \leq 0.$$

- Observe that $\frac{\partial H}{\partial t} = (e^t - 1)(\ddot{\varphi}_t + \dot{\varphi}_t) - b'(t)$.
- Now $\dot{\varphi}_t + \varphi_t = \log\left(\frac{\omega_t^n / V_t}{\omega_X^n}\right) - h(x)$ and $\omega_t = e^{-t}(\omega_0 - \eta) + \eta + dd^c\varphi_t$.
- Thus $\ddot{\varphi}_t + \dot{\varphi}_t = \Delta_t(\dot{\varphi}_t) + \mathrm{Tr}_{\omega_t}(e^{-t}(\eta - \omega_0)) - \frac{d \log V_t}{dt}$, while

  $\Delta_t H = (e^t - 1)\Delta_t(\dot{\varphi}_t) - \Delta_t(\varphi_t) = (e^t - 1)\Delta_t(\dot{\varphi}_t) - n + \mathrm{Tr}_{\omega_t}(\eta + e^{-t}(\omega_0 - \eta))$.

  $\Rightarrow \left(\frac{\partial}{\partial t} - \Delta_t\right) H = -\mathrm{Tr}_{\omega_t}(\omega_0) - (e^t - 1)\frac{d \log V_t}{dt} - b'(t) + n = -\mathrm{Tr}_{\omega_t}(\omega_0)$. □

Conclusion

Assume either $T_{max} = +\infty$ or $T_{max} < +\infty$ and $V_{T_{max}} > 0$.

- We have shown $\frac{\omega_t^n}{V_t} = f_t \omega_X^n$ with $f_t = e^{\dot{\varphi}_t + \varphi_t + h} \leq C_2$.
- By [Guo-Phong-Song-Sturm 24] we obtain for all $0 \leq t < T_{max}$,
  - $\mathrm{diam}(X, \omega_t) \leq D_0$;
  - $c_\varepsilon r^{2n+\varepsilon} V_t \leq \mathrm{Vol}_{\omega_t}(B_{\omega_t}(x, r))$ for all $x \in X$ and $0 < \varepsilon, 0 < r < D$;
  - $V_t \leq V_0$ since $V_t \xrightarrow{t \to T_{max}} V_\infty$.

Theorem (Gromov)

The metric spaces (X, d_{ω_t}) are relatively compact in the G-H sense.

Problem

What is the Gromov-Hausdorff limit of (X, d_{ω_t}) as $t \to T_{max}$?

Minimal model of general type

Assume X is a minimal model of general type, i.e.

- K_X is *nef* $\Longleftrightarrow T_{max} = +\infty$ (X smooth minimal model),
- and K_X is *big* $\Longleftrightarrow V_{max} > 0 \Longleftrightarrow \nu(X) = n$.

Theorem (Birkar-Cascini-Hacon-McKernan 10)

There exists a holomorphic birational map $f : X \longrightarrow X_{can}$, where

- *X_{can} is a mildly singular projective variety (canonical model of X);*
- *the canonical bundle $K_{X_{can}} > 0$ is ample.*

Theorem (Eyssidieux-G-Zeriahi 09)

There exists a unique Kähler-Einstein current T_{KE} on X_{can}, i.e.

- *a Kähler form on X_{can}^{reg} such that $\mathrm{Ric}(T_{KE}) = -T_{KE}$;*
- *T_{KE} has bounded local potentials near X_{can}^{sing}.*

Minimal model of general type (end)

Theorem (Cao 85 / Tsuji 88 / Tian-Zhang 06 / Song 14 / Wang 18)

Assume X min model of general type, fix ω_0 an arbitray Kähler form on X. The Normalized Kähler-Ricci Flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t$$

with initial datum ω_0 exists for all times $t > 0$ and

- *weakly converges, as $t \to +\infty$, to the Kähler-Einstein current $f^* T_{KE}$;*
- *convergence is smooth on the ample locus $Amp(K_X) = f^{-1}(X_{can}^{reg})$;*
- *the diameter (X, d_{ω_t}) is uniformly bounded along the flow;*
- *(X, d_{ω_t}) converges in the Gromov-Hausdorff topology to (X_{can}, d_{KE}).*

$\hookrightarrow$ Problem: extend this to non smooth minimal models.

Intermediate Kodaira dimension

Assume X is a smooth, abundant, intermediate minimal model, i.e.

- K_X is *semi-ample* : $K_X = f^*L$ where $f : X \to Y$ with $L > 0$ on Y,
- and $1 \leq \nu(X) \leq n-1$. In particular K_X is *nef* ($T_{max} = +\infty$).
- Set $Y^0 = Y \setminus \left(Y^{sing} \cup \mathrm{Singv}(f)\right)$; fibers X_y are smooth CY if $y \in Y^0$.

Theorem (Song-Tian 12)

There exists a twisted Kähler-Einstein current T_{KE}, i.e.

- *T_{KE} is a Kähler form on $f^{-1}(Y^0)$ with $\mathrm{Ric}(T_{KE}) = -T_{KE} + \omega_{WP}$;*
- *$\omega_{WP} \geq 0$ is a Weil-Petersson type metric;*
- *T_{KE} has bounded local potentials near $f^{-1}(Y^0)$.*

- Partial ext. to sing min models [EGZ 18]. Regularity theory of T_{KE} ?
- Problem: get rid of abundance assumption.

Intermediate Kodaira dimension (end)

Theorem (Song-Tian 12 / Song-Tian-Zhang 19 / Hein-Lee-Tosatti 24)

Assume X is a smooth intermediate abundant minimal model. Fix ω_0 an arbitrary Kähler form on X. The Normalized Kähler-Ricci Flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) - \omega_t$$

with initial datum ω_0 exists for all times $t > 0$ and

- *weakly converges, as $t \to +\infty$, to the Kähler-Einstein current T_{KE};*
- *the convergence is smooth on $f^{-1}(Y^0)$;*
- *(X, d_{ω_t}) converges in the G-H topology to (Y, d_{KE}) if $\nu(X) = 1$.*

$\hookrightarrow$ Pbm: extend this to $\nu(X) \neq 1$ and non smooth minimal models.

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