Kähler Green's functions

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Introductory workshop

Quasi-subharmonic functions

- Let (X, ω) be a compact Kähler manifold of complex dimension n.

Definition

- *A function $v : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-subharmonic if it is locally the sum of a subharmonic and a smooth function.*
- *It is called ω-subharmonic if $(\omega + dd^c v) \wedge \omega^{n-1} \geq 0$. Equivalently*

$$\Delta_\omega v := n \frac{dd^c v \wedge \omega^{n-1}}{\omega^n} \geq -n.$$

- *We let $SH(X, \omega)$ denote the set of all ω-subharmonic functions.*

- Goal: study properties of the map $\omega \longmapsto SH(X, \omega)$.
- Warning: $(\omega, v) \mapsto (\omega + dd^c v) \wedge \omega^{n-1}$ affine in v but non-linear in ω !

Kähler Green's function

Definition (Green's function)

Given ω Kähler form we consider $G^\omega \in \mathcal{C}^\infty(X \times X \setminus \mathrm{Diag}, \mathbb{R})$ s.t.

- $G^\omega(x, y) = G^\omega(y, x)$ *for all* $(x, y) \in X \times Y$;
- $G^\omega(x, y) \sim -\frac{1}{[d_\omega(x,y)]^{2n-2}}$ *if* $n \geq 2$;
- $y \mapsto G^\omega_x(y) = G^\omega(x, y) \in SH(X, \omega)$ *with*

$$\frac{1}{V_\omega}(\omega + dd^c G^\omega_x) \wedge \omega^{n-1} = \delta_x \Longleftrightarrow \Delta_\omega G^\omega_x = n\left\{V_\omega \delta_x - \omega^n\right\},$$

  where $V_\omega = \int_X \omega^n$ *and* δ_x*=Dirac mass at point* x;
- $y \mapsto G^\omega_x(y)$ *is normalized by* $\int_X G^\omega_x(y)\, \omega^n(y) = 0$.

- Classical: there exists a unique solution, the Green's function.
- Problem: study how $\omega \mapsto G^\omega(x, y)$ varies, uniformly wrt (x, y).

Key estimates

- Fix ω_X a reference Kähler form normalized by $\int_X \omega_X^n = 1$.
- Fix $A, B > 0$ and $p > 1$. Set $f_\omega = V_\omega^{-1}\omega^n/\omega_X^n$ and consider

$$\mathcal{K}(X,p,A,B) := \left\{\omega \text{ Kähler s.t. } \int_X \omega \wedge \omega_X^{n-1} \leq A \text{ and } \int_X f_\omega^p \omega_X^n \leq B\right\}.$$

Theorem (Guo-Phong-Song-Sturm 24 / G-Tô 24 / Vu 24)

Fix $r < \frac{n}{n-1}$ and $s < \frac{2n}{2n-1}$. Then for all $x \in X$ and $\omega \in \mathcal{K}(X,p,A,B)$,

- $\sup_{y \in X} G_x^\omega \leq C_0 = C_0(n,p,A,B)$;
- $\int_X |G_x^\omega|^r \frac{\omega^n}{V_\omega} \leq C_1 = C_1(n,p,r,A,B)$;
- $\int_X |\nabla G_x^\omega|^s \frac{\omega^n}{V_\omega} \leq C_2 = C_2(n,p,s,A,B)$.

- Goal of Lecture 2: proof of these uniform estimates.

Green's formula for ω-subharmonic functions

- Assume $(\omega + dd^c v) \wedge \omega^{n-1} \geq 0$ with $\int_X v\omega^n = 0$. Then

$$v(x) \;=\; \int_X v \frac{(\omega + dd^c G_x^{\omega}) \wedge \omega^{n-1}}{V_\omega}$$

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$$
\begin{aligned}
v(x) &= \int_X v \frac{(\omega + dd^c G_x^\omega) \wedge \omega^{n-1}}{V_\omega} = \int_X v \frac{dd^c G_x^\omega \wedge \omega^{n-1}}{V_\omega} \\
&= \int_X G_x^\omega \frac{dd^c v \wedge \omega^{n-1}}{V_\omega}
\end{aligned}
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- Thus $v(x) \leq \sup_X G_x^\omega \leq C_0$.
- By Hölder inequality and symmetry $G_x^\omega(y) = G_y^\omega(x)$, we also obtain

$$\int_X |v|^r \frac{\omega^n}{V_\omega} \leq C_1 \quad \text{and} \quad \int_X |\nabla v|^s \frac{\omega^n}{V_\omega} \leq C_2.$$

- Thus proving key estimates for G_x^ω or arbitrary v is the same.

Application 1: Diameter bounds

Corollary

Under previous assumptions $\mathrm{diam}(X,\omega) \leq D = 2C_2(n, p, 1, A, B)$.

- Fix $(a, b) \in X^2$ such that $d_\omega(a, b) = \mathrm{diam}(X, \omega)$.
- The function $\rho : x \in X \mapsto d_\omega(a, x) \in \mathbb{R}^+$ is 1-Lipschitz with $\rho(a) = 0$.
- Thus $0 = V_\omega \rho(a) = \int_X \rho(\omega + dd^c G_a^\omega) \wedge \omega^{n-1}$ yields, by Stokes,

$$\int_X \rho \omega^n = \int_X d\rho \wedge d^c G_a^\omega \wedge \omega^{n-1}$$

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- Similarly $V_\omega \rho(b) = \int_X \rho\,\omega^n + \int_X \rho dd^c G_b^\omega \wedge \omega^{n-1}$ hence

$\mathrm{diam}(X,\omega) = \rho(b)$

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$$\mathrm{diam}(X,\omega) = \rho(b) \leq \int_X |\nabla G_a^\omega|_\omega \frac{\omega^n}{V_\omega} + \int_X |\nabla G_b^\omega|_\omega \frac{\omega^n}{V_\omega} \leq 2C_2.$$

□

Application 2: Non collapsing

Corollary

Under previous assumptions $\frac{\mathrm{Vol}_\omega(B_\omega(x,r))}{V_\omega} \geq c_\delta r^{2n+\delta}$ *for* $0 < r < D, x \in X$.

- Fix $0 \leq \chi \leq 1$ with $\chi \equiv 1$ on $B_\omega(x, r/2)$ and $\chi \equiv 0$ off $B_\omega(x, r)$.
- As $|\nabla \chi|_\omega \leq \frac{6}{r}$ the function $\rho\chi$ is 7-Lipschitz, where $\rho(y) = d_\omega(x, y)$.
- Fix $0 < s < \frac{2n}{2n-1}$, $s^* =$conj exp.

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- Fix $0 < s < \frac{2n}{2n-1}$, s^* =conj exp. By Green's formula at $y \notin B_\omega(x, r)$

$$\int \rho\chi\omega^n = \int d(\rho\chi) \wedge d^c G_y^\omega \wedge \omega^{n-1} \leq C_2(s) V_\omega^{\frac{1}{s}} \mathrm{Vol}_\omega(B_\omega(x, r))^{\frac{1}{s^*}}.$$

- Applying now Green's formula at $z \in \partial B_\omega(x, r/2)$ we obtain

$$\frac{r}{2} = \int \rho\chi \frac{\omega^n}{V_\omega} - \int d(\rho\chi) \wedge d^c G_z^\omega \wedge \frac{\omega^{n-1}}{V_\omega}$$

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- The conclusion follows since $s^* = 2n + \delta \in (2n, +\infty)$. $\square$

Application 3: Uniform Sobolev inequalities

Theorem

Fix $1 < q < \frac{2n}{n-1}$ *and* $\omega \in \mathcal{K}(X, p, A, B)$. *For all* $u \in W^{1,2}(X)$, *we have*

$$\left(\frac{1}{V_\omega}\int_X |u-\overline{u}|^{2q}\omega^n\right)^{1/r} \leq C_S \frac{1}{V_\omega}\int_X |\nabla u|^2_\omega \omega^n,$$

where $\overline{u} = \frac{1}{V_\omega}\int_X u\omega^n$ *and* $C_S = C_S(n, p, q, A, B) > 0$.

- Set $\mathcal{G}_x^\omega = G_x^\omega - C_0 - 1$. We show later $\frac{1}{V_\omega}\int_X \frac{dG_x^\omega \wedge d^c G_x^\omega \wedge \omega^{n-1}}{(-\mathcal{G}_x^\omega)^{1+\beta}} \leq \frac{1}{\beta}$.
- Green's formula and Hölder inequality yield

$$|u(x) - \bar{u}| \leq \frac{1}{\beta^{1/2}}\left(\frac{1}{V_\omega}\int_X (-\mathcal{G}_x^\omega)^{1+\beta}|\nabla u|^2_\omega \omega^n\right)^{1/2}.$$

- Conclude by Minkowski's inequality+main estimate for gradient. □

Condition on the cohomology class

- The first condition $\int_X \omega \wedge \omega_X^{n-1} \leq A$ is cohomological.
- It is equivalent to the fact that $\{\omega\} \in B(R_A) \subset H^{1,1}(X, \mathbb{R})$.
- By $\partial\overline{\partial}$-lemma $\omega = \theta + dd^c \varphi_\omega$ with $-C_A \omega_X \leq \theta \leq C_A \omega_X$.
- Volume $V_\omega = \int_X \omega^n = \{\omega\}^n$ can collapse but no blowup $V_\omega \leq C_A^n$.

Example

- *Assume* $X = \mathbb{P}^1 \times \mathbb{P}^1$ *is the product of two Riemann spheres, endowed with the Kähler form* $\omega_\lambda(x, y) = \lambda \omega_{\mathbb{P}^1}(x) + \lambda^{-1} \omega_{\mathbb{P}^1}(y)$, *where* $\lambda > 0$.
- *Note* $\omega_\lambda^2 = 2\omega_{\mathbb{P}^1}(x) \wedge \omega_{\mathbb{P}^1}(y) = 2\omega_X^2$, *hence* $f_\lambda \equiv 2$, *2nd condition OK.*
- *Moreover volumes* $V_{\omega_\lambda} = \int_X \omega_\lambda^2 = \int_X 2\omega_{\mathbb{P}^1}(x) \wedge \omega_{\mathbb{P}^1}(y) = 2$ *are constant, while* $\mathrm{diam}(X, \omega_\lambda) \sim \lambda \to \infty$ *as* $\lambda \to \infty$.

Optimal condition on the density

- Similar results by [GPPS24] when $\int_X f_\omega(\log[7+f_\omega])^p \omega_X^n \leq B$, $p > n$;
- [G-Guenancia-Zeriahi 23] extend these to the quasi-optimal condition

$$(*)_p \quad \int_X f_\omega(\log[7+f_\omega])^n(\log\log[7+f_\omega])^p \omega_X^n \leq B, \text{ with } p > 2n.$$

- Compare [Kolodziej 98]: $(*)_p \Longrightarrow \mathrm{Osc}_X(\varphi_\omega) \leq M_B$ if $p > n$.

Example

- *Consider* $\omega = dd^c \chi \circ L$, χ *convex increasing,* $L(z) = \log|z|$ *in* $\mathbb{C}^n$.
- *Then* $\omega^n = f_\omega dV_{eucl}$ *with* $f_\omega \sim \frac{\chi'' \circ L\,(\chi' \circ L)^{n-1}}{|z|^{2n}}$.
- *For* $\chi(t) = (\log(-t))^{-1}$ *we obtain* $\mathrm{diam}(\mathbb{B}^n, \omega) = +\infty$,

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- *For* $\chi(t) = (\log(-t))^{-1}$ *we obtain* $\mathrm{diam}(\mathbb{B}^n, \omega) = +\infty$, *while*

$$(*)_p \text{ satisfied by } f_\omega \Longleftrightarrow p < 2n-1.$$

Quasi-plurisubharmonic projection

Definition

- *A function* $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ *is quasi-plurisubharmonic if it is locally the sum of a plurisubharmonic and a smooth function.*
- *It is called* ω*-plurisubharmonic if* $\omega + dd^c\varphi \geq 0$.
- $PSH(X, \omega)$ *denotes the set of all* ω*-plurisubharmonic functions.*

- Key tool: a priori estimates for solutions to cplx MA equations.
- Lower bound: if v is ω-sh then $\varphi = P_\omega(v) \leq v$ where

$$P_\omega(v) := \sup\{u \in PSH(X, \omega)\ u \leq v\} \in PSH(X, \omega)$$

satisfies a complex Monge-Ampère equation associated to $\Delta_\omega v$.
- We actually use a twisted version of this rough idea.

Twisted complex Monge-Ampère equations

Proposition (G-Tô 24)

Let v (resp. φ) be a bounded ω-sh (resp. ω-psh) function such that

$$(\omega + dd^c v) \wedge \omega^{n-1} = e^{tv} g \omega^n \quad \text{and} \quad (\omega + dd^c \varphi)^n \geq e^{nt\varphi} g^n \omega^n,$$

where $t > 0$, $p > n$ and $0 \leq g \in L^p(\omega^n)$. Then $\varphi \leq v$.

- Definition : u is a ω-sh subsolution if $(\omega + dd^c u) \wedge \omega^{n-1} \geq e^{tu} g \omega^n$.
- Max pple+balayage: v is the envelope of bounded ω-sh subsolutions.
- The AM-GM inequality ensures that $(\omega + dd^c \varphi) \wedge \omega^{n-1} \geq e^{t\varphi} g \omega^n$.
- This allows one to conclude since $PSH(X, \omega) \subset SH(X, \omega)$. $\square$
- Application to follow: if $p > n$ then v is uniformly bounded below.

Exponential integrability of ω-psh functions

Theorem

Fix $A, B > 0$ and $p > 1$. There exists $\alpha = \alpha(n, p, A, B) > 0$ such that for all $\omega \in \mathcal{K}(X, p, A, B)$ and $\varphi \in PSH(X, \omega)$ with $\sup_X \varphi = 0$,

$$\int_X \exp(-\alpha\varphi)\omega_X^n \leq C,$$

where $C = C(\alpha, n, p, A, B) > 0$ is independent of ω, φ.

- [Skoda 72]: establishes exponential integrability of psh functions.
- [Tian 87]: uses α-invariant to study $\exists$ of K-E metrics (ω fixed).
- [Demailly-Kollar 01]: relate α-invariants and log can thresholds.
- [Zeriahi 01]: very general uniform versions of Skoda's result.
- Thm follows from [Z 01], $\omega = \theta + dd^c\varphi$ and $\varphi - P_\theta(0)$ bounded.

Uniform a priori estimates for MA potentials

Theorem (Kolodziej 98 ... Di Nezza-G-Guenancia 23)

Fix $p > 1$, $A, B > 0$ and $\omega \in \mathcal{K}(X, p, A, B)$. Assume that there exists $\varphi \in PSH(X, \omega) \cap L^\infty(X)$, $p' > 1$ and $B' > 0$ s.t. $\int_X g^{p'} \omega_X^n \leq B'$ and

$$\frac{1}{V_\omega}(\omega + dd^c\varphi)^n = g\omega_X^n.$$

Then $\mathrm{Osc}_X(\varphi) \leq C = C(n, p, p', A, B, B')$.

- This is the key a priori estimate for everything that follows.
- Goes back to [Yau78], [Kolodziej 98], [Eyssidieux-G-Z 09], [EGZ08], [Demailly-Pali 10]. More recently [G-Lu 21], [Guo-Phong-Tong 23].
- Follows from previous thm + general L^∞ a priori estimates [DNGG23].

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Bounding ω-sh functions from above 1

Lemma A

Fix $p>1$, $A,B>0$ and $\omega \in \mathcal{K}(X,p,A,B)$. Fix $a>0$ and let v be a quasi-sh function on X such that $\Delta_\omega v \geq -a$ and $\int_X v\omega^n = 0$. Then

$$\sup_X v \leq C_1 \left[a + \frac{1}{V_\omega} \int_X |v| \omega^n \right],$$

where $C_1 = C_1(n,p,A,B) > 0$ is independent of v and ω.

- Statement and assumptions are homogeneous of degree 1, wlog $a = n$.
- Set $v_+ = \max(v,0)$ and consider $\varphi \in PSH(X,\omega)$ bounded solution of
$$(\omega + dd^c\varphi)^n = \frac{1+v_+}{1+M}\omega^n,$$
with $\sup_X \varphi = -1$, where $M = \int_X v_+ \frac{\omega^n}{V_\omega} = \frac{1}{2}\int_X |v| \frac{\omega^n}{V_\omega}$.
- GOAL: φ bounded below and $v_+ \lesssim (-\varphi)^\alpha$, with $\alpha = \frac{n}{n+1}$.

Bounding ω-sh functions from above 2

- Set $H = 1 + v_+ - \varepsilon(-\varphi)^\alpha$, where $\alpha = \frac{n}{n+1}$ and $\frac{\varepsilon^{n+1}\alpha^n}{(1+\alpha\varepsilon)^n} = 1 + M$.
- As $-dd^c(-\varphi)^\alpha = \alpha(1-\alpha)(-\varphi)^{\alpha-2} d\varphi \wedge d^c\varphi + \alpha(-\varphi)^{\alpha-1} dd^c\varphi$, get

$$\Delta_\omega(-\varepsilon(-\varphi)^\alpha) \geq \alpha\varepsilon(-\varphi)^{\alpha-1}\Delta_\omega\varphi \overset{AM-GM}{\geq} n\alpha\varepsilon(-\varphi)^{\alpha-1}\left[\left(\frac{1+v_+}{1+M}\right)^{\frac{1}{n}} - 1\right].$$

- Therefore $\Delta_\omega H \geq -n + n\alpha\varepsilon(-\varphi)^{\alpha-1}\left[\left(\frac{1+v_+}{1+M}\right)^{\frac{1}{n}} - 1\right]$.
- Using $(-\varphi)^{1-\alpha} \geq 1$, we get at x_0 such that $H(x_0) = H_{max}$,

$$(1+\alpha\varepsilon)(-\varphi)^{1-\alpha} \geq (-\varphi)^{1-\alpha} + \alpha\varepsilon \geq \alpha\varepsilon\left(\frac{1+v_+}{1+M}\right)^{\frac{1}{n}}.$$

- Thus $\varepsilon(-\varphi)^\alpha = \varepsilon(-\varphi)^{n(1-\alpha)} \geq \frac{\alpha^n\varepsilon^{n+1}}{(1+\alpha\varepsilon)^n}\frac{1+v_+}{1+M}$

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- Therefore $\Delta_\omega H \geq -n + n\alpha\varepsilon(-\varphi)^{\alpha-1}\left[\left(\frac{1+v_+}{1+M}\right)^{\frac{1}{n}} - 1\right]$.
- Using $(-\varphi)^{1-\alpha} \geq 1$, we get at x_0 such that $H(x_0) = H_{max}$,

$$(1+\alpha\varepsilon)(-\varphi)^{1-\alpha} \geq (-\varphi)^{1-\alpha} + \alpha\varepsilon \geq \alpha\varepsilon\left(\frac{1+v_+}{1+M}\right)^{\frac{1}{n}}.$$

- Thus $\varepsilon(-\varphi)^\alpha = \varepsilon(-\varphi)^{n(1-\alpha)} \geq \frac{\alpha^n\varepsilon^{n+1}}{(1+\alpha\varepsilon)^n}\frac{1+v_+}{1+M} = 1 + v_+$, i.e. $H \leq 0$.

Bounding ω-sh functions from above 3

- Note that $\varepsilon \leq c_n(1+M)$ since $\frac{\varepsilon^{n+1}\alpha^n}{(1+\alpha\varepsilon)^n} = 1+M$.
- Thus $\frac{(\omega+dd^c\varphi)^n}{V_\omega} = FdV_X$ with $F = \frac{1+v_+}{1+M} f$

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- Since $\int_X f^p dV_X \leq A$, can fix $1 < r < p$ and use Hölder to obtain

$$\int_X F^r dV_X \leq c_n^r \int_X (-\varphi)^{r\alpha} f^r dV_X \leq \left(\int_X f^p dV_X\right)^{\frac{r}{p}} \left(\int_X (-\varphi)^{\frac{rp\alpha}{p-r}} dV_X\right)^{\frac{p-r}{p}}.$$

- Integrals uniformly bounded by assumption+Skoda-Zeriahi's result.
- Thus φ bounded and $\sup_X v \leq \sup_X v_+ \leq \varepsilon(-\varphi)^\alpha \leq c_n[1+M]C_0$.
- Conclusion follows since $M = \frac{1}{2V_\omega}\int_X |v|\omega^n$. $\square$

Functions with bounded Laplacian 1

Lemma B

Fix $p > 1$, $A, B > 0$ and $\omega \in \mathcal{K}(X, p, A, B)$. Let u be a continuous function such that $\int_X u\omega^n = 0$ and $||\Delta_\omega u||_{L^\infty(X)} \leq 1$. Then

$$||u||_{L^\infty(X)} \leq C_2,$$

where $C_2 = C_2(n, p, A, B) > 0$ is independent of u and ω.

- By Lemma A suffices to show $M = \frac{1}{V_\omega} \int_X |u|\omega^n \leq C_2$. Wlog $M \geq 1$.
- $\boxed{\text{Claim : } C_2 = 8C_0(1 + 4n^2 C_1^2)^2 \text{ ok}}$, C_0 from Thm MA with $g = 2^n$.
- Pbm homogeneous of deg 1, wlog $||\Delta_\omega u||_{L^\infty(X)} \leq \delta = \frac{1}{4(1+4n^2 C_1^2)^2}$.

  $\hookrightarrow$ we are going to show that $\boxed{M \leq 2C_0}$.

Functions with bounded Laplacian 2

- Set $H = n\Delta_\omega u$, $\varepsilon = 1/M$ and $t = \sqrt{\delta} = \frac{1}{2(1+4n^2C_1^2)}$.
- Let $\psi \in PSH(X,\omega) \cap L^\infty(X)$ be the unique bounded ω-psh solution of

$$(\omega + dd^c\psi)^n = e^{nt\varepsilon(\psi-u)}(1+H)^n\omega^n.$$

- AM-GM inequality yields $(\omega + dd^c\psi) \wedge \omega^{n-1} \geq e^{t\varepsilon(\psi-u)}(1+H)\omega^n$.
- [G-Tô 24] $\Rightarrow \psi \leq u$ since $(\omega + dd^c u) \wedge \omega^{n-1} = e^{t\varepsilon(u-u)}(1+H)\omega^n$.
- Now $\omega_\psi^n \leq 2^n\omega^n$ as $H \leq n\delta \leq 1$ hence [DNGG23] $\Rightarrow \mathrm{Osc}_X(\psi) \leq C_0$.
- Mass controls+normalization yield $\varepsilon \sup_X \psi \geq -\delta/t - 4nC_1^2 t = -\frac{1}{2}$.
- This yields $u \geq (\psi - \sup_X \psi) + \sup_X \psi$

Functions with bounded Laplacian 2

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- This yields $u \geq (\psi - \sup_X \psi) + \sup_X \psi \geq -C_0 - \frac{M}{2}$.
- By symmetry $u \leq \frac{M}{2} + C_0$ hence $M = \int_X |u| \frac{\omega^n}{V_\omega} \leq \frac{M}{2} + C_0$. $\square$

Control of the supremum of G_x^ω

Corollary A

Fix $p > 1$, $A, B > 0$ and $\omega \in \mathcal{K}(X, p, A, B)$. Then for all $x \in X$,

$$\int_X |G_x^\omega| \frac{\omega^n}{V_\omega} \leq C_0 \ \text{ and } \ \sup_{y \in X} G_x^\omega(y) \leq C_0 = C_0(n, p, A, B).$$

- Set $h = -\mathbf{1}_{\{G_x \leq 0\}} + \int_{\{G_x \leq 0\}} \frac{\omega^n}{V_\omega}$. Note $-1 \leq h \leq 1$ and $\int_X h\omega^n = 0$.
- For $\Delta_\omega v = h$ with $\int_X v\omega^n = 0$, Lemma B yields $||v||_{L^\infty(X)} \leq C$.
- Thus $C \geq v(x) = \frac{1}{V_\omega} \int_X v(\omega + dd^c G_x) \wedge \omega^{n-1}$

$$= \frac{1}{V_\omega} \int_X G_x dd^c v \wedge \omega^{n-1} = n \int_{\{G_x \leq 0\}} (-G_x) \frac{\omega^n}{V_\omega}.$$

- Since $\int_X G_x \omega^n = 0$, we infer $\int_X |G_x| \frac{\omega^n}{V_\omega} = 2 \int_{\{G_x \leq 0\}} (-G_x) \frac{\omega^n}{V_\omega} \leq \frac{2C}{n}$.
- It therefore follows from Lemma A that $\sup_X G_x \leq C_0$. $\square$

Control of the L^r-norm of G_x^ω

Corollary B

Fix $p > 1$, $A, B > 0$, $\omega \in \mathcal{K}(X, p, A, B)$ and $1 \le r < \frac{n}{n-1}$. For all $x \in X$,

$$\frac{1}{V_\omega} \int_X |G_x^\omega|^r \omega^n \le C_1 = C_1(n, p, r, A, B).$$

- Set $\mathcal{G}_x = G_x - C_0 - 1 \le -1$ and consider u the ω-sh solution of $\frac{1}{V_\omega}(\omega + dd^c u) \wedge \omega^{n-1} = \frac{(-\mathcal{G}_x)^\beta \omega^n}{\int_X (-\mathcal{G}_x)^\beta \omega^n}$, with $\int_X u\omega^n = 0$, $0 < \beta < \frac{1}{n}$.
- Since $1 \le -\mathcal{G}_x$ we have $\int_X (-\mathcal{G}_x)^\beta \frac{\omega^n}{V_\omega} \le \int_X (-\mathcal{G}_x) \frac{\omega^n}{V_\omega} = 1 + C_0$.
- We are going to show that $u \ge -C$ is uniformly bounded below. Thus

$$-C \le u(x) = \int_X \mathcal{G}_x \frac{(\omega + dd^c u) \wedge \omega^{n-1}}{V_\omega} = -\frac{\int_X (-\mathcal{G}_x)^{1+\beta} \frac{\omega^n}{V_\omega}}{\int_X (-\mathcal{G}_x)^\beta \frac{\omega^n}{V_\omega}}.$$

- Therefore $\int_X (-\mathcal{G}_x)^{1+\beta} \frac{\omega^n}{V_\omega} \le C[1 + C_0]$

Control of the L^r-norm of G_x^ω

Corollary B

Fix $p>1$, $A,B>0$, $\omega \in \mathcal{K}(X,p,A,B)$ and $1 \le r < \frac{n}{n-1}$. For all $x \in X$,

$$\frac{1}{V_\omega}\int_X |G_x^\omega|^r \omega^n \le C_1 = C_1(n,p,r,A,B).$$

- Set $\mathcal{G}_x = G_x - C_0 - 1 \le -1$ and consider u the ω-sh solution of $\frac{1}{V_\omega}(\omega + dd^c u)\wedge\omega^{n-1} = \frac{(-\mathcal{G}_x)^\beta \omega^n}{\int_X (-\mathcal{G}_x)^\beta \omega^n}$, with $\int_X u\omega^n = 0$, $0<\beta<\frac{1}{n}$.
- Since $1 \le -\mathcal{G}_x$ we have $\int_X (-\mathcal{G}_x)^\beta \frac{\omega^n}{V_\omega} \le \int_X (-\mathcal{G}_x)\frac{\omega^n}{V_\omega} = 1 + C_0$.
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- Therefore $\int_X (-\mathcal{G}_x)^{1+\beta}\frac{\omega^n}{V_\omega} \le C[1+C_0] \Longrightarrow$ OK for $r < 1 + \frac{1}{n}$.

Control of the L^r-norm: bounding u from below

- Consider the solution $\varphi \in PSH(X,\omega) \cap L^\infty(X)$, $\sup_X \varphi = 0$, of

$$\frac{1}{V_\omega}(\omega + dd^c\varphi)^n = \frac{(-\mathcal{G}_x)^{n\beta}\omega^n}{\int_X (-\mathcal{G}_x)^{n\beta}\omega^n}.$$

- The density of the RHS is bounded from above by $(-\mathcal{G}_x)^{n\beta} f_\omega$.
- Hölder $\int_X (-\mathcal{G}_x)^{n\beta p'} f_\omega^{p'} dV_X \leq \left(\int_X f_\omega^p dV_X\right)^{\frac{p'-1}{p-1}} \left(\int_X (-\mathcal{G}_x)^{n\beta p' s'} \frac{\omega^n}{V_\omega}\right)^{\frac{1}{s'}} \leq A'$.
- OK if we choose $p' > 1$ very close to 1, and $s' = \frac{p-1}{p-p'}$ (close to 1) is the conjugate exponent of $s = \frac{p-1}{p'-1}$, so that $n\beta p' s' < 1$+Corollary A.
- Theorem MA shows $\varphi \geq -M_0$ and AM-GM yields $u \geq \varphi / C' \geq -C$.
- Recursive argument L^r control OK for $r < 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{n}{n-1}$ $\square$.

The weighted gradient

- Although $\nabla G_x^\omega \notin L^2$, the following weighted version holds:

Lemma C

Fix $\beta > 0$. *Then* $\frac{1}{V_\omega} \int_X \frac{dG_x^\omega \wedge d^c G_x^\omega \wedge \omega^{n-1}}{(-G_x^\omega + C_0 + 1)^{1+\beta}} \leq \frac{1}{\beta}$.

- Consider $u(y) = (-G_x^\omega(y) + C_0 + 1)^{-\beta}$, with $u(x) = 0$.
- Note that $1 \leq -G_x^\omega + C_0 + 1$ hence $0 \leq u \leq 1$.
- Since $du = \frac{\beta dG_x^\omega}{(-G_x^\omega + C_0 + 1)^{1+\beta}}$ and $0 = \frac{1}{V_\omega} \int_X u(\omega + dd^c G_x^\omega) \wedge \omega^{n-1}$,

  we obtain $\frac{\beta}{V_\omega} \int_X \frac{dG_x^\omega \wedge d^c G_x^\omega \wedge \omega^{n-1}}{(-G_x^\omega + C_0 + 1)^{\beta+1}} = \frac{1}{V_\omega} \int_X u\omega^n \leq 1$. $\square$

Control of the gradient

Corollary C

Fix $p > 1$, $A, B > 0$, $\omega \in \mathcal{K}(X, p, A, B)$ and $0 < s < \frac{2n}{2n-1}$. For all $x \in X$,

$$\frac{1}{V_\omega} \int_X |\nabla G_x^\omega|^s \omega^n \leq C_2 = C_2(n, p, s, A, B).$$

- Fix $s < \frac{2n}{2n-1}$, $0 < \beta$ very small and $r = \frac{s}{2-s}(1 + \beta) < \frac{n}{n-1}$.
- Set $2\alpha = s(1 + \beta)$ and $\mathcal{G}_x = G_x - C_0 - 1$. Lemma C and Hölder yield

$$\int_X |\nabla G_x|^s \omega^n = \quad = \quad \int_X \frac{|\nabla \mathcal{G}_x|^s}{|\mathcal{G}_x|^\alpha} |\mathcal{G}_x|^\alpha \omega^n$$

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$$\begin{aligned} \int_X |\nabla G_x|^s \omega^n = \quad &= \int_X \frac{|\nabla \mathcal{G}_x|^s}{|\mathcal{G}_x|^\alpha} |\mathcal{G}_x|^\alpha \omega^n \\ &\leq \left(\int_X \frac{|\nabla \mathcal{G}_x|^2}{|\mathcal{G}_x|^{\frac{2\alpha}{s}}} \omega^n \right)^{\frac{s}{2}} \left(\int_X |\mathcal{G}_x|^{\frac{2\alpha}{2-s}} \omega^n \right)^{\frac{2-s}{2}} \end{aligned}$$

Control of the gradient

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$$\begin{aligned}
\int_X |\nabla G_x|^s \omega^n = &= \int_X \frac{|\nabla \mathcal{G}_x|^s}{|\mathcal{G}_x|^\alpha} |\mathcal{G}_x|^\alpha \omega^n \\
&\leq \left(\int_X \frac{|\nabla \mathcal{G}_x|^2}{|\mathcal{G}_x|^{\frac{2\alpha}{s}}} \omega^n \right)^{\frac{s}{2}} \left(\int_X |\mathcal{G}_x|^{\frac{2\alpha}{2-s}} \omega^n \right)^{\frac{2-s}{2}} \\
&= \left(\int_X \frac{|\nabla \mathcal{G}_x|^2}{|\mathcal{G}_x|^{1+\beta}} \omega^n \right)^{\frac{s}{2}} \left(\int_X |\mathcal{G}_x|^r \omega^n \right)^{\frac{2-s}{2}}
\end{aligned}$$

Control of the gradient

Corollary C

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&= \left(\int_X \frac{|\nabla \mathcal{G}_x|^2}{|\mathcal{G}_x|^{1+\beta}} \omega^n \right)^{\frac{s}{2}} \left(\int_X |\mathcal{G}_x|^r \omega^n \right)^{\frac{2-s}{2}} \leq C_2 V_\omega. \quad \square
\end{aligned}$$

Further results

Theorem (G-Tô 24)

The estimates do not depend on the choice of complex structure.

Theorem (Guo-Phong-Song-Sturm 24 / Vu 24)

Most estimates are valid for singular varieties.

Theorem (Li 21)

- *Fix* $\mathcal{A} \subset H^{1,1}(X, \mathbb{R})$ *a compact subset of the Kähler cone.*
- *Fix* $B > 0$ *and* $p > 1$. *Set* $f_\omega = V_\omega^{-1}\omega^n/\omega_X^n$ *and consider*

$$\mathcal{K}(p, \mathcal{A}, B) := \left\{\omega \text{ Kähler form s.t. } \{\omega\} \in \mathcal{A} \text{ and } \int_X f_\omega^p \omega_X^n \leq B\right\}.$$

There exists $\alpha, C > 0$ *such that* $d_\omega \leq C d_{\omega_X}^\alpha$ *for all* $\omega \in \mathcal{K}(p, \mathcal{A}, B)$.

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