SCIENTIFIC PROGRAM FOR SLMATH WORKSHOP
LIFE AFTER THE TELESCOPE CONJECTURE
DECEMBER 9-13, 2024

0.1. **Talk 1 Monday: History of the Telescope Conjecture.**

- Speaker: Douglas Ravenel
- Abstract: In this talk, the speaker will explain the origins of, and motivation for, the Telescope Conjecture.
- Suggested references: [Rav84, Bar20, Rav87, Mah82, Mah81, MRS01, Mil81, BBB$^+$21]

0.2. **Talk 2 Monday: Overview.**

- Speaker: Tomer Schlank
- Abstract: An overview of the week and of the disproof, outlining all the pieces we will need to develop and how they fit together. Alternatively, the speaker may discuss one of several height 1 proofs.
- Suggested references: [BHLS23] or [Mil81, Mah82]

0.3. **Talk 3 Monday: Cyclotomic spectra.**

- Speaker: Maxine Calle
- Abstract: The fundamental work of Dundas–Goodwillie–McCarthy relates the algebraic K-theory of a connective ring spectrum R to its topological cyclic homology $\mathrm{TC}(R)$, which is in turn a functor of the cyclotomic spectrum $\mathrm{THH}(R)$.

  In this talk, the speaker will introduce a modern definition of the category of cyclotomic spectra, due to Nikolaus and Scholze. A basic example is given by $\mathrm{THH}(R)$ when R is a ring spectrum. The speaker will introduce several invariants of cyclotomic spectra, namely TP, TC^-, TC, and TR. They will also briefly mention THH with coefficients in a bimodule, and the related formalism of p-polygonic spectra. Finally, they will give the statement of the Dundas–Goodwillie–McCarthy theorem.
- Suggested references: [NS18], [DGM13], [KMN23]

0.4. **Talk 4 Monday: Chromatically localized algebraic K-theory.**

- Speaker: Liam Keenan
- Abstract: In several ways, the algebraic K-theory of a height n ring simplifies after localization at a telescope $T(n+1)$. For us, the most fundamental will be Land–Mathew–Meier–Tamme purity, which is intimately tied to Clausen–Mathew–Naumann–Noel descent. The speaker will explain the purity theorem from [LMMT20], which states that

  $$L_{T(n+1)}\mathrm{K}(R) \simeq L_{T(n+1)}\mathrm{K}(L_{T(n)\oplus T(n+1)}R),$$

  and may sketch a few ingredients of the proof. The most basic example of purity is Mitchell's theorem, which states that $L_{T(n+1)}\mathrm{K}(R) = 0$ whenever

R is a discrete ring and $n \geq 1$; the speaker will note how this simplifies the use of the Dundas–Goodwillie–McCarthy theorem.
- Suggested references: [LMMT20], [CMNN20]

0.5. **Talk 1 Tuesday : Cyclotomic boundedness.**
- Speaker: Hana Jia Kong
- Abstract: Hesselholt–Madsen proved the Lichtenbaum–Quillen conjecture for p-adic local fields K (with $p > 2$) by proving the stronger statement that $V(1)_*\mathrm{TR}(\mathcal{O}_K)$ is bounded.

  Bounded TR is now best interpreted as boundedness in the Antieau–Nikolaus t-structure. The speaker will introduce this t-structure, and note that $\mathrm{THH}(\mathbb{F}_p)$ is bounded. The main aim of the talk should be to characterize cyclotomic boundedness in more concrete terms, as the combination of the Segal conjecture and canonical vanishing. Bounded cyclotomic rings admit a Bökstedt class, and the speaker will discuss its basic properties.
- Suggested references: Sections 2.2, 2.3 and 2.4 of [AN21]

0.6. **Talk 2 Tuesday: Boundedness of the Adams summand.**
- Speaker: Christian Ausoni
- Abstract: This talk will explain the theorem, by Ausoni–Rognes, that $V(2)_*\mathrm{TR}(\ell)$ is bounded for primes $p > 3$. (The speaker may assume that $p > 5$, so that $V(2)$ exists as a homotopy commutative and associative ring, or alternatively may make use of the motivic spectral sequence.) Here, $\ell = \mathrm{BP}\langle 1\rangle$ is the Adams summand of p-local complex K-theory. This was the first example of a higher height Lichtenbaum–Quillen theorem, and the speaker will also explain how to deduce chromatic redshift.

  The computations going into the Ausoni–Rognes result will be explained explicitly enough that they may be adapted to compute $V(2)_*\,\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ on Thursday (speakers may want to coordinate these two talks).
- Suggested references: [AR02], [HRW22] and [BHLS23, §6 & 7]

0.7. **Talk 3 Tuesday: Trace methods beyond connective rings.**
- Speaker: Kate Ponto
- Abstract: This talk will explain how many of the results comparing K-theory and TC can be extended beyond the setting of connective rings, and in particular to -1-connective rings, because the universal localizing invariant of -1-connective rings are built out of those of connective rings. In the case of rings that are fixed points of $\mathbb{Z}$-actions on connective rings, this talk will explain how this can be accomplished using the work of Land–Tamme on the K-theory of pullbacks.
- Notes for the speaker:
  - The most important thing to cover is [Lev22, Theorem B], and the corresponding parts of Land–Tamme that go into the proof of it. This will be applied in the example where E is the fiber of the map from K to TC, and $R \to S$ is the map $\ell \to \pi_0\ell$ (where $\pi_0\ell$ denotes the Eilenberg–MacLane spectrum $\mathbb{Z}_{(p)}$). The $\mathbb{Z}$-action is by Adams operations on ℓ, and is trivial on $\pi_0\ell$.
  - More generally, there is a version that works for arbitrary -1-connective rings [Lev22, Theorem C], which you might want to discuss depending

on available time. It might be worth noting that the relevant property used to prove these results happens at the level of localizing motive in the sense of Blumberg–Gepner–Tabuada [BGT13].

- Suggested references: [LT19], [Lev22]

0.8. **Talk 4 Tuesday: Cochains on a circle.**

- Speaker: Cary Malkiewich
- Abstract: This talk will study the cochains on the circle as an $\mathbb{E}_\infty$-ring in cyclotomic spectra. In particular, its study is largely controlled by studying the free loop space of the p-adic circle $B\mathbb{Z}_p$. As a consequence, the coassembly map for the TC of the fixed points by a trivial $\mathbb{Z}$-action is usually not an isomorphism.
- Suggested references: [BHLS23, §3]

0.9. **Talk 1 Wednesday: Ambidexterity and cyclotomic extensions.**

- Speaker: Allen Yuan
- Abstract: This talk will present the property of higher semi-additivity in general and in particular for $Sp_{T(n)}$. In addition higher semi-additivity will be used to defined the telescopic cyclotomic extensions. The notion of cyclotomic completion will be discussed.
- Notes for the speaker:
  - state ambidexterity
  - construct the cyclotomic extensions using ambidexterity
  - define and discuss cyclotomically complete spectra.
- Suggested references: [CSY22], [CSY21] and [BCSY24]

0.10. **Talk 2 Wednesday: Cyclotomic redshift.**

- Speaker: Shay Ben-Moshe
- Abstract: In this talk we prove that $T(n+1)$-localized algebraic K-theory satisfies descent for π-finite p-group actions on stable ∞-categories of chromatic height up to n. As a consequence, we use this to show that cyclotomic extensions "redshift" and to present cyclotomic completion of a $T(n+1)$-localized algebraic K-theory as an co-assembly map.
- Suggested references: [BCSY23]

0.11. **Talk 1 Thursday: Calculating** TC **of** $\ell^{hp^k\mathbb{Z}}$**.**

- Speaker: David Lee
- Abstract: This talk will disprove the telescope conjecture at height 2 and primes $p > 5$, by a direct computational approach. Specifically, the speaker will prove that, for k sufficiently large and $\mathbb{Z}$ acting by the Adams operation Ψ^{p+1},
$$V(2)_* \operatorname{TC}(\ell^{hp^k\mathbb{Z}})$$
has non-finite homotopy groups.
- Notes for the speaker: Some key points to emphasize:
  - Using the module structure over $\operatorname{TC}^-(\mathbb{S}^{B\mathbb{Z}})$, the homotopy fixed point spectral sequence computing $V(2)_* \operatorname{TC}^-(\ell^{hp^k\mathbb{Z}})$ may be viewed as a family of spectral sequences indexed over $p^k\mathbb{Z}_p$. At $0 \in p^k\mathbb{Z}_p$, the spectral sequence is the S^1 homotopy fixed point spectral sequence for $\operatorname{TC}^-(\ell)^{hp^k\mathbb{Z}}$. Accounting for algebra structure, only finitely many

differentials control this homotopy fixed point spectral sequeence, and therefore the 0 fiber controls a neighborhood of $0 \in p\mathbb{Z}_p$.
  - TP can be handled in a similar manner, and we can control φ and can using these same techniques.
  - The key non-formal step is controlling the class ζ in π_{-1}.
- Suggested references: Oberwolfach report 34/2023, [BHLS23, §7], final video/notes from Mark Behrens' graduate course

0.12. **Talk 2 Thursday: Asymptotic constancy I.**

- Speaker: Eva Belmont
- Abstract: The key input to seeing that the $T(n)$-local TC doesn't commute with taking -fixed points in the case of interest is asymptotic constancy. Asymptotic constancy allows one to reduce checking this to the case of a trivial -action. This talk and the next will be dedicated to showing asymptotic constancy for the THH of the ring spectra of interest. The key properties of the ring spectra that are used to show this are that the Z-actions on them are *locally unipotent*, they satisfy the height n Lichtnebaum–Quillen property, and are almost compact.
- Notes for the speaker: This and the following talk form a pair and the speakers should coordinate to divide the following material over two talks:
  - Statements of asymptotic constancy
  - locally unipotent actions
  - A.3 and 4.1
  - The Dehn twist trivialization.
  - Bootstrapping from constancy for THH as a spectrum to constancy for THH as a cyclotomic spectrum.
- Suggested references: [BHLS23, §4.1,4.2,A.3]

0.13. **Talk 3 Thursday: Asymptotic constancy II.**

- Speaker: Ishan Levy
- Abstract: This is a continuation of the previous talk. Namely, it will be explained how to bootstrap asymptotic constancy at the level of spectra to the level of cyclotomic spectra, and how this can be used to deduce constancy at the level of T(n+1)-homology.
- References [BHLS23, §4.3,4.4]

0.14. **Talk 4 Thursday: Q&A Session.** This will be an hour reserved for the audience to ask questions.

0.15. **Talk 1 Friday: Examples at low primes and large heights.**

- Speaker: Andrew Senger
- Abstract: To apply asymptotic constancy, we require fp-type n ring spectra that both satisfy Lichtenbaum–Quillen and admit locally unipotent $\mathbb{Z}$-actions. For our study of the telescope conjecture, these $\mathbb{Z}$-actions should be related to cyclotomic extensions.

  At primes $p \geq 5$ and height $n+1 = 2$, we may use the Adams summand $\mathrm{BP}\langle 1\rangle$, the computations of Ausoni–Rognes, and geometrically defined Adams operations. At primes $p < 5$ and height $n+1 = 2$, we may still use geometrically defined Adams operations on the Adams summand, but need a replacement for the Ausoni–Rognes proof of Lichtenbaum–Quillen.

At a general prime and height, this talk will summarize how $\mathrm{BP}\langle n\rangle$ can be constructed as an $\mathbb{E}_3$-ring with $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-Adams operations, and how one may prove the Lichtenbaum–Quillen property for it.
- Suggested references: [BHLS23, §5,§7]

0.16. **Talk 2 Friday: Assembling the disproof.**
- Speaker: Jeremy Hahn
- Abstract: TBA
- Suggested references:

0.17. **Talk 3 Friday: Future directions.**
- Speaker: Lior Yanovski
- Abstract: TBA
  - using this to get growth in stable stems
- Suggested references:

0.18. **Talk 4 Friday: Future directions.**
- Speaker: Robert Burklund
- Abstract: TBA
  - Galois lifting and the universality of hyperdescent as an obstruction
- Suggested references:

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[AR02] Christian Ausoni and John Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. **188** (2002), no. 1, 1–39. MR 1947457

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