Knot complements decomposing into prisms

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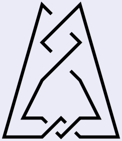
Hyperbolic knots

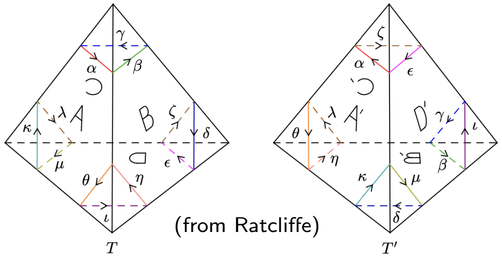
Definition

A knot K is *hyperbolic* if $\mathbb{S}^3 - K$ admits a *hyperbolic structure*: a complete, finite-volume Riemannian metric with sectional curvature $\equiv -1$.

Mostow-Prasad rigidity: a hyperbolic structure, if it exists, is unique.

Example (Thurston, late 1970's): the figure-eight

$\mathbb{S}^3 -$  $\cong$



(from Ratcliffe)

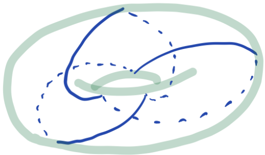
Geometrization for knot complements in $\mathbb{S}^3$

Theorem (Thurston, early 1980's)

Every prime knot is either a torus knot, a satellite knot, or hyperbolic.

A *torus knot*

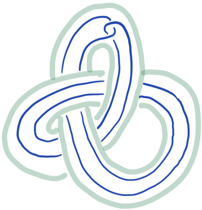
One that lies on a copy of $\mathbb{S}^1 \times \mathbb{S}^1$ "standardly embedded" in $\mathbb{S}^3$.



Right: the trefoil is a torus knot.

A *satellite knot*

A "knot within a (nontrivial) knot".



Right: a satellite of the trefoil.

From *The next 350 million knots*, B. Burton.

The knot tables

# xings	# hyperbolic knots	# non-hyp.	*credit*
≤ 10	243	7	*classical (+Perko '74)*
11	551	1	*Conway (+Caudron) '70s*
12	2,176	0	
13	9,985	3	*Dowker-Th. '80s*
14	46,969	3	
15	253,285	8	
16	1,388,694	11	*Hoste-Weeks-Th. '96*
17	8,053,363	30	
18	48,266,380	86	
19	294,130,212	246	*Burton '20*
20	1,847,318,507	921	*Thistlethwaite '25*
Total	2,199,470,365	1,316	

~~Open~~ Question: As crossing $\# \to \infty$, does the ratio $\frac{\#\text{hyp. knots}}{\#\text{ all knots}} \to 1$?

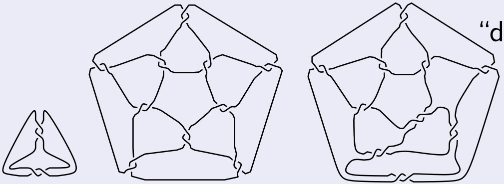
A vague motivating question

How many knot complements decompose into "nice" polyhedra?

Example: three "platonic" knots

Meaning their complements decompose into *regular* ideal polyhedra in $\mathbb{H}^3$.

figure-eight



"dodecahedral knots"

(Aitchison-Rubinstein '90)

Theorem (Reid, '91) The figure-eight is the only arithmetic knot.

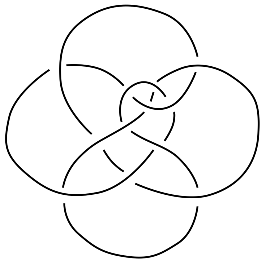
Theorem (Hoffman, '14) There are no other dodecahedral knots.

Corollary There are no other platonic knots.

Another "nice" example

The "Boyd knot" *12n706* is "mixed-platonic"

Its complement decomposes into regular ideal tetrahedra *and* octahedra.



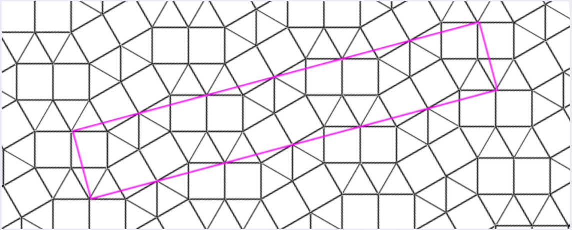


Figure: On the right: the peripheral tiling of $\mathbb{S}^3 - 12n706$.

Open Question: do there exist more mixed-platonic knots?

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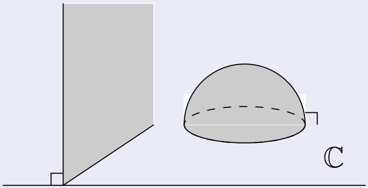
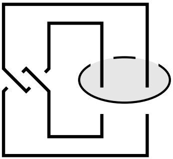
Totally geodesic surfaces

Definition

A smoothly embedded surface S in a hyperbolic 3-manifold M is *totally geodesic* if every point of S has a chart neighborhood $\phi\colon U \to \mathbb{H}^3$ such that $\phi(S \cap U) = \Pi \cap \phi(U)$, where $\Pi \subset \mathbb{H}^3$ is a totally geodesic plane.

Geodesic planes in $\mathbb{H}^3$





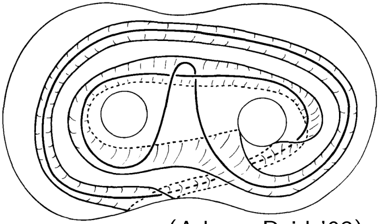
An example of totally geodesic surfaces "in nature"

C. Adams, '85: *every properly embedded three-punctured sphere in a hyperbolic 3-manifold is properly isotopic to totally geodesic.*

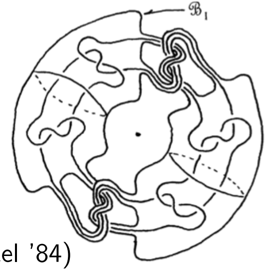
Topological obstructions to total geodesicity

A properly embedded surface S in a hyperbolic 3-manifold M is *not* properly isotopic to totally geodesic if any of the following are present:

- A **compressing disk** $D \subset M$ with $D \cap S = \partial D$.
- A **compressing annulus** $A \subset M$ with $A \cap S = \partial A$
- An **accidental parabolic**: an annulus $A \subset M$ with $\partial A = \gamma_1 \sqcup \gamma_2$, where $\gamma_1 = A \cap S$, and γ_2 lies on a cusp cross-section.



(Adams-Reid '93)



(Oertel '84)

The Menasco-Reid conjecture

Menasco-Reid '92: *No hyperbolic knot complement in* $\mathbb{S}^3$ *contains a closed, embedded totally geodesic surface.*

The Menasco-Reid conjecture

Menasco-Reid '92: *No hyperbolic knot complement in $\mathbb{S}^3$ contains a closed, embedded totally geodesic surface.*

Classes of knots known to satisfy the conjecture.

- Two-bridge knots (Hatcher-Thurston '79)
- Alternating knots. (Menasco '84)
- Montesinos knots (Oertel '84)
- Knots of braid index 3 (Lozano-Przytycki '85) and 4 (Matsuda '02)
- Almost alternating knots (Adams et. al. '92)
- Toroidally alternating knots (Adams '94)
- Three-bridge and double torus knots (Ichihara-Ozawa '00)

These all possess *topological* obstructions to a(ny) totally geodesic surface.

Also: many $\leq$ 15-crossing census knots (Basilio–Lee–Malionek '24)

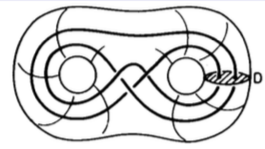
Totally geodesic surfaces are *improbable*.

...even in the absence of topological obstruction.

Fact. Suppose $S \subset M \doteq \mathbb{S}^3 - K$ is a closed, embedded, surface with no compressing disks or annuli, nor accidental parabolics. Then:

1. each component of $M \backslash\backslash S$ admits a unique hyperbolic structure with totally geodesic boundary, and
2. S is totally geodesic in M if and only if:
   1. the geodesic boundary components of $M \backslash\backslash S$ are isometric; and
   2. that isometry is realized by the gluing that recovers M.

An example from Adams-Reid '93



vol ≈ 12.046

$+$

vol ≈ 6.452

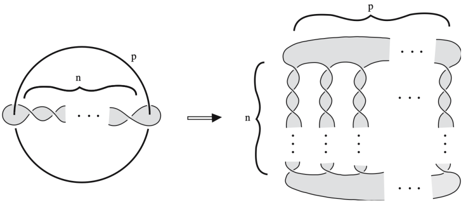
$=$

vol ≈ 21.219

In the other direction

Adams-Schoenfeld '05

Totally geodesic *Seifert* surfaces.



Leininger '06

Closed surfaces with arbitrarily small (non-zero) principal curvatures.

Closed *immersed* (non-embedded) totally geodesic sfces

- Reid '91: in the figure-8 knot complement
- Aitchison-Rubinstein '97: in the dodecahedral knot complements.

Other notable results

Non-closed immersed surfaces

- Agol: the only non-embedded 3-punct spheres are in twist knot comps
- Le-Palmer '23: two-punctured torus in $\mathbb{S}^3 - 7_4$.

Tangetially Related Open Question

Is there a knot complement with a *separating*, embedded non-closed totally geodesic surface.

An arithmeticity criterion

Bader–Fisher–Miller–Stover '21: if a hyperbolic 3-manifold contains infinitely many immersed totally geodesic surfaces then it is arithmetic.

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The basics

A **hidden symmetry** of a space X is a homeomorphism between finite-degree covers of X that does not lift a symmetry of X.

Which hyperbolic manifolds have hidden symmetries?

Lots

A characterization of arithmeticitiy

Fact: A hyperbolic 3-manifold is arithmetic if and only if it has infinitely many hidden symmetries.

Hidden symmetries for the rest of us

Fact: A non-arithmetic hyperbolic 3-manifold has a hidden symmetry if and only if it non-normally covers an orbifold.

Where do hidden symmetries come from, 1

Non-compact arithmetic hyperbolic 3-manifolds

Fact: $M = \mathbb{H}^3/\Gamma$ non-compact, orientable, and finite-volume is *arithmetic* if and only if $\Gamma < \mathrm{PGL}_2(\mathcal{O}_d)$ (up to conjugacy) for some square-free integer $d > 1$, where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$.

Elements of $\mathrm{PGL}_2(\mathbb{Q}(\sqrt{-d})) \rightsquigarrow$ hidden symmetries of such M.

A related exercise

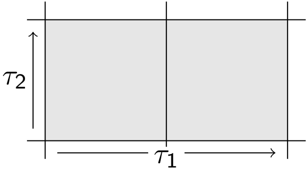
For an arbitrary $M \in \mathrm{SL}_2(\mathbb{Q})$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $ad - bc = 1$,

show that you can choose $n \in \mathbb{N}$ so that for any

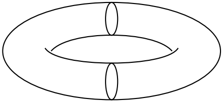
$$N \in \ker\left(\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})\right), \ N = \begin{pmatrix} xn+1 & yn \\ zn & wn+1 \end{pmatrix},$$

the conjugate MNM^{-1} still belongs to $\mathrm{SL}_2(\mathbb{Z})$.

Where do hidden symmetries come from, 2: a toy example



$T = \mathbb{R}^2/\langle \tau_1, \tau_2 \rangle$



No self-isometry of T exchanges the geodesics corresponding to τ_1 and τ_2.

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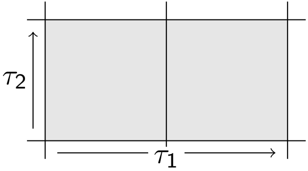

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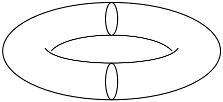

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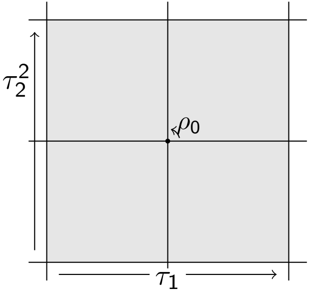
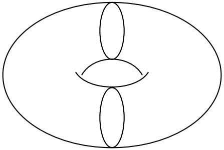


$\widetilde{T} = \mathbb{R}^2/\langle \tau_1, \tau_2^2 \rangle$

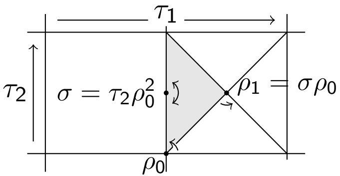


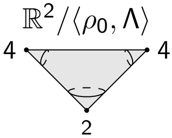
(2x thicker torus)

But there is such a self-isometry of $\widetilde{T} = \mathbb{R}^2/\langle \tau_1, \tau_2^2 \rangle$.

Toy example, continued: looking *down* from T

Observe: $\Lambda = \langle \tau_1, \tau_2 \rangle$ is a non-normal subgroup of $\langle \rho_0, \Lambda \rangle$,
the full symmetry group of a tiling of $\mathbb{R}^2$ by squares.





Fact/Definition (orientable setting)

The *rigid Euclidean orbifolds* are $\mathbb{S}^2(2,4,4)$, $\mathbb{S}^2(3,3,3)$, and $\mathbb{S}^2(2,3,6)$.
These are respectively quotients of tilings of $\mathbb{R}^2$ by
right triangles, equilateral triangles, and their barycentric subdivisions.

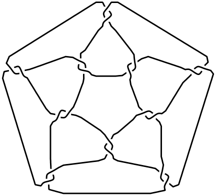
Definition. A *rigid-cusped* hyperbolic 3-orbifold is one whose cusp cross-sections are rigid Euclidean orbifolds.

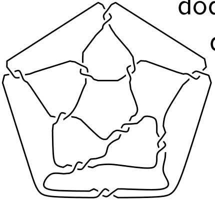
Three knot complements with hidden symmetries

Proposition (Neumann-Reid, *Topology '90*)

A hyperbolic knot complement has hidden symmetries
$\Leftrightarrow$ *it covers a rigid-cusped orbifold.*

$\mathbb{S}^3 - 4_1$ decomposes into regular ideal tetrahedra

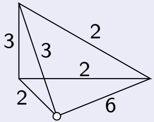


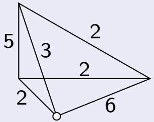


dodec knot comps decompose into regular ideal dodecahedra

Barycentric subdivision $\rightsquigarrow$ tetrahedral orbifolds

Covered by $\mathbb{S}^3 - 4_1$





Covered by the dodec. knot complements

Knot complements with hidden symmetries (??)

Question (Neumann-Reid, Topology '90)

Are there any other hyperbolic knot complements with hidden symmetries?

Conjecture (Neumann–Reid, *Kirby Problem List* '95)

There are no more hyperbolic knot complements with hidden symmetries.

Recall additional context

- Reid, '91: the fig-8 is the only knot with arithmetic complement.
- Reid + Hoffman '14: the only platonic knot complements are those of 4_1 and the two dodecahedral knots

Hyperbolic knot complements *without* hidden symmetries

- All two-bridge knots, except 4_1. (Reid–Walsh, 2008)
- The hyperbolic $(-2, 3, n)$-pretzel knots. (Macasieb–Mattmann)
- Knots obtained by surgery on the Berge manifold. (Hoffman)
- All 313,210 with ≤ 15 crossings, except 4_1. (SnapPy/Sage+ϵ)
- The $(i_1, \ldots, i_n)$-pretzel knots, $n \geq 5$ odd, $i_j \geq N(n)$ for all j, with i_1 even and i_j odd for $j > 1$. (Millichap)
- All but finitely many fillings on one cusp of any $\leq$ 9-crossing link complement. (Chesebro-D-Mondal)
- All those of "highly twisted" knots in the sense of Purcell. (Hoffman–Millichap–Worden)
- All knot complements covering manifolds of volume $< 6v_0$, for $v_0 = 1.0198...$ (Chesebro-D-Hoffman–Millichap–Mondal–Worden)
- All knot complements covering a filling of one cusp of $\mathbb{S}^3 - 6^2_2$, except $\mathbb{S}^3 - 4_1$ (Chesebro-D-Hoffman–Millichap–Mondal–Worden)

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Overview and strategy

Joint with Arshia Gharagozlou (my former PhD student) & Neil Hoffman.

Emerged from an AIM SQuaRE with (besides myself):
Michelle Chu, Eric Chesebro, Neil, Priyadip Mondal, and Genevieve Walsh

The Big Idea

Rather than look *among knot complements* for those with

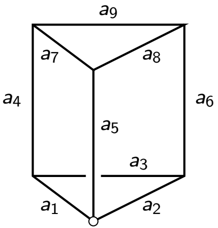
- closed embedded totally geodesic surfaces
- and/or hidden symmetries;

look *for knot complements*
among manifolds known to have these properties.

Our target class

Manifold covers of the "prism orbifolds".

Prism orbifolds: a rich class of rigid-cusped orbifolds



Hyperbolic Coxeter Prisms (HCP)

A 9-tuple $(a_1, \ldots, a_9)$ with $a_i \in \{2, 3, \ldots\}$
$\rightsquigarrow$ prism $\subset \mathbb{H}^3$ with angles $\frac{\pi}{a_1}, \ldots, \frac{\pi}{a_9}$.

Circled vertex is ideal $\Leftrightarrow$ up to permutation,

$$(a_1, a_2, a_5) = (2, 3, 6),\ (3, 3, 3), \text{ or } (2, 4, 4).$$

Fact/Dfn: reflections in the faces of a HCP determine a *prism orbifold*.

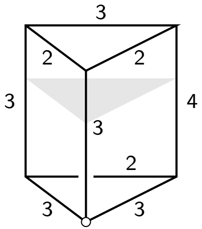
Foundation: Lakeland-Roth '20

Classified prisms with exactly one ideal vertex (Andreev's Thrm criteria).

- $(2, 3, 6)$-vertex: eight infinite families, 32 specific configurations
- $(3, 3, 3)$-vertex: 22 specific configurations
- $(2, 4, 4)$-vertex: four infinite families, 24 specific configurations

Totally geodesic surfaces

Fact. Every prism contains a compact totally geodesic triangle that meets each of its three rectangular faces at a right angle.



Consequence.

Every finite-degree manifold cover of a prism orbifold contains a closed, embedded, totally geodesic surface.

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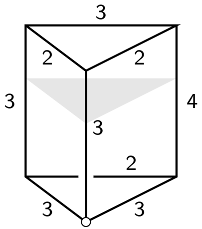


Consequence.

Every finite-degree manifold cover of a prism orbifold contains a closed, embedded, totally geodesic surface.

The upshot

Any knot complement covering a prism orbifold would be:

1. a counterexample to the Neumann-Reid conjecture; *and*
2. a counterexample to the Menasco-Reid conjecture.

Finding knots, step 1: pre-filtering

Test necessary conditions for an orbifold to have a knot complement cover; rule out prism orbifolds that fail one.

Conditions from the work of Hoffman

- No $(2, 4, 4)$-cusps. (2022)
- Triviality under cusp-killing (2014)
- $(2, 3, 6)$-cusp double covered by $(3, 3, 3)$ (2022)
- Algebraic integer entries $\Rightarrow$ unit meridian (2014)

Tools: hand computation, "house-made" Python scripts,
Sage (Computer Algebra System)

Result: twelve infinite families and 78 specific configurations
reduces to two infinite subfamilies and 12 specific configurations.

Zoom in: cusp killing

A characterization of knot complements

Fact. A one-cusped hyperbolic manifold M is a knot complement in $\mathbb{S}^3$ if and only if $\pi_1 M = \langle\langle \alpha \rangle\rangle$ for some peripheral element α.

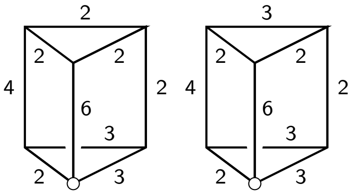
Consequence: If an orbifold $O = \mathbb{H}^3/\Gamma$ is covered by a hyperbolic knot complement in $\mathbb{S}^3$ then $\Gamma = \langle\langle \Lambda \rangle\rangle$, where $\Lambda < \Gamma$ is a peripheral subgroup.

Checking this criterion on the isotropy graph (one-skeleton).

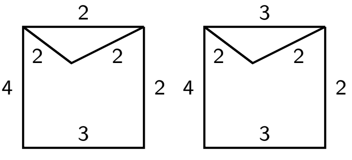
1. Erase the edges corresponding to peripheral torsion.
2. For any degree 2 vertices in the resulting graph, relabel the edges connected to it by the gcd of their labels. If the gcd is 1, prune these two edges; if not, resolve the vertex to be part of an edge or loop.
3. Prune any edges incident to a degree 1 vertex, and remove any vertices not incident to any edge.

Repeat until either the graph is empty or it cannot be resolved further.

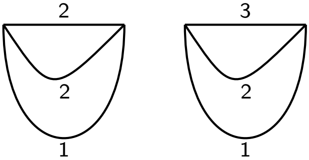
The algorithm in action: two examples



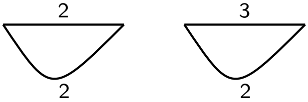
(a) At the outset



(b) Torsion edges at the cusp removed



(c) Vertices with valence 2 resolved

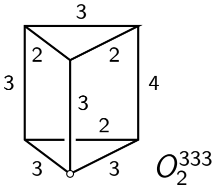


(d) The edges labeled 1 removed

The upshot: the left-hand orbifold *fails*, and on the right *passes*, the cusp-killing test.

Step 2: enumerate covers

Fact: lcm(vertex group orders) divides the degree of any manifold cover.



The results of hand computation:

The minimum such lcm, over all remaining *orientable* prism orbifolds, is 24.

Attained on $\widetilde{O}_2^{333}$, $\widetilde{O}_3^{333}$, $\widetilde{O}_4^{333}$, $\widetilde{O}_{19}^{236}$.

Key tool for subgroup enumeration: the `low_index` Python module

- Given a group presentation & $k \in \mathbb{N}$, returns all subgroups up to index k, encoded by permutation representations.
- Coded by Culler w/help from Dunfield & Goerner, 2022
- Implements an algorithm of C.C. Sims (comp. grp. theory, mid-90's)

Result:

- $32,425$ degree ≤ 24 covers of $\widetilde{O}_2^{333}$, $29,432$ of $\widetilde{O}_3^{333}$,
- and $306,552$ of $\widetilde{O}_4^{333}$.

Step 3: *filter* the covers

Fact. Given a face-pairing presentation, a permutation representation for a subgroup directly determines a cell decomposition of the associated cover.

The Fact $\rightsquigarrow$ custom-coded Python scripts

...testing the following necessary conditions to be a knot complement.

- torsion-free/manifold covers;
- with *one cusp*;
- and first homology $\cong \mathbb{Z}$.

Result:

- **20** one-cusped manifold covers of $\widetilde{O}_2^{333}$ with $H_1 \cong \mathbb{Z}$,
- **22** of $\widetilde{O}_3^{333}$, and
- **12** of $\widetilde{O}_4^{333}$.

These are candidates to be knot complements.

Step 4: so are any of them knot complements?

Answer: No.

Step 4: so are any of them knot complements?

Answer: No. But...

Procedure, run on each remaining manifold cover.

- Convert cell decomp to a triangulation by triangulating the prism.
- Simplify to a one-vertex (hence ideal) triangulation using Regina's `intelligentSimplify`
- Pass the triangulation to SnapPy, search for finite cyclic fillings
- If found, use `filled_triangulation` to produce & simplify a triangulation for it.
- Pass back to Regina, apply lens space recognition.

Result: for each of $i = 2$ and 3...

- One cover $M_{i,1}$ of $\widetilde{O}_i^{333}$ is a knot complement in $L(13, 3)$.
- One cover $M_{i,2}$ of $\widetilde{O}_i^{333}$ is a knot complement in $L(22, 5)$.

So we get knots

Theorem (D-Gharagozlou–Hoffman)

At least four distinct knot complements in $\mathbb{S}^3$ cover a "prism orbifold".

Proof.

For $i \in \{2, 3\}$ and $j \in \{1, 2\}$, the preimage $\widetilde{M}_{i,j}$ of $M_{i,j}$ in the universal cover $\mathbb{S}^3$ of $L(13, 3)/L(22, 5)$ is a knot complement.

(Follows from Gonzales-Acuna–Whitten.) □

Some things we know about these knots.

- Their **volumes**: ≈ 585.55 (first two), ≈ 990.86 (second two)
- ...hence lower bounds on crossing number: 159 (first two), 270 (second two) via Adams.
- Chirality: yes.

Challenge (a thing we can't do): draw a picture of one of these knots.