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A 2-categorical approach to dg schemes

dg schemes: "geometric objects" whose local coordinate rings are dg algebras

We will work only over k, char $k = 0$

A dg-algebra will mean a commutative unital dg-algebra over k.

Explicitly: • $A = \bigoplus_{i \leq 0} A^i$ graded algebra + $a \cdot b = (-1)^{\deg a \deg b}\, b a$

• A is equipped with a differential $d : A \to A$ of degree 1 ($d^2 = 0$)

so that $d(ab) = (da)b + (-1)^{\deg a} a(db)$

Example: Let $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ be polynomials defining an affine scheme $V \subset \mathbb{A}^n$.

Consider the dga $(A^{\cdot}, d)$

where

$$A^0 = k[x_0, \dots, x_n]$$

$$A^{-1} = A^0 \xi_1 \oplus \dots \oplus A^0 \xi_r$$

$$A^{-p} = \bigoplus_{i_1 < \dots < i_p} A^0 \xi_{i_1} \cdots \xi_{i_p}$$

i.e.

$$A^\bullet = \Lambda^\bullet_{A^0} A^{-1} \quad \text{- alternating algebra}$$

Define

$$d\xi_i = f_i \in A^0 \quad \text{+ extend using Leibnitz}$$

Then $(A^\bullet, d)$ is a dg algebra which is just the Koszul complex of $f_1, \dots, f_r \in A^0$.

<u>Note</u>: • If we pass to cohomology $\Rightarrow$ get a graded algebra $h^\bullet(A)$ over $h^0(A) = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$

• $f_1, \dots, f_r$ is a regular sequence $\Leftrightarrow$ $\dim V = n-r$ ($\Rightarrow V$ lci)

$\Leftrightarrow$ $A^\bullet \to h^0(A)$ - resolution

$\Leftrightarrow$ $A^\bullet \to h^0(A)$ - qis

<u>Recall</u>: A morphism $A \to B$ of dga is a quasi-isomorphism if

$$h^\bullet(A) \to h^\bullet(B)$$

is an isomorphism.

Heuristically if $A \to h^0(A)$ is a qis then we should think of $V = \mathrm{Spec}\, h^0(A)$ as the dg-scheme defined by A.

<u>Goal</u>: generalize the above heuristics to general dga (not necessarily formal)

<u>Note</u>: The Koszul complex is an example of a quasi-free dga i.e. $A^\natural = (A \text{ without } d)$ is a graded commutative unital k-algebra on the generators $x_1, \ldots, x_n$ (deg 0) $\xi_1, \ldots, \xi_r$ (deg -1)

<u>Example</u>: If we look at a standard affine open $\{g \neq 0\} \subset V$ for some $g(x) \in k[x_1, \ldots, x_n]$, then we can adjoin two new variables

y (deg $y = 0$)

ξ (deg $\xi = -1$)

with the extra relation $d\xi = yg - 1$ then the resulting algebra

$$k[x_1, \ldots, x_n, y]\{\xi_1, \ldots, \xi_r, \xi\} / (d\xi_i = f_i,\ d\xi = yg - 1)$$

is again quasi free

In other words: localization along g in the dg world preserves freeness.

Example: (derived intersection)

$$\begin{tikzcd} X \text{ smooth} \arrow[d, leftrightarrow] & & B \\ V \arrow[r, hook] & \mathbb{A}^n \qquad h^0(A) & A^0 \arrow[l] \arrow[u] \end{tikzcd}$$

smooth
regular

affine coordinate rings

Now in dg world replace $h^0(A)$ by a q.is to it: $A^\bullet$.

Since $A^\bullet$ is quasi free ($\Rightarrow$ free as an A_0 module)

$\Rightarrow$

$$h^i(A^\bullet \otimes_{A^0} B) = \mathrm{Tor}^{A^0}_{-i}(h^0(A), B)$$

dg-philosophy: fibered products of dg schemes should give derived tensor products

$$\left[\Rightarrow [V]\cdot[X] = td\,(T_{\mathbb{A}^n} - T_V - T_X)^{-1} \cap \tau\left(Tor_\bullet^{\mathcal{O}_{\mathbb{A}^n}}(\mathcal{O}_V, \mathcal{O}_X)\right) \right]$$

<u>Goal</u>: Construct a category of dg schemes so that

- dg algebras define dg schemes
- every dg scheme has locally associated with it certain dgas, well defined up to qis
- fiber products of dg schemes are locally given by derived tensor products
- dg schemes can be glued together where qis are considered as isos

<u>Example</u>: Let U_1, U_2 two dg-schemes. Let

$U_{1,g} \subset U_1$ - open defined by $g \neq 0$
$U_{2,h} \subset U_2$ - open defined by $h \neq 0$

Assume that we are given qi's

$$U_{1,g} \xleftarrow{qi's} V \xrightarrow{qi's} U_{2,g}$$

then we should be able to glue U_1 and U_2 along V.

Explicitly if $U_1 \leftrightarrow$ dga A_1
$g \in A_1$
$U_2 \leftrightarrow$ dga A_2
$h \in A_2$

then

$$\begin{tikzcd} & & B & & \\ A_1 \arrow[r] & \dfrac{A_1[y]\{y\}}{dy = 1-yg} \arrow[ur, "qi's"] & & \dfrac{A_2[y]\{y\}}{dy = 1-yh} \arrow[ul, "qi's"'] & A_2 \arrow[l] \end{tikzcd}$$

should be a gluing diagram.

- derived moduli spaces should satisfy UMP in this category so that the dg structure on the moduli space should be determined by the UMP (universal mapping property).

<u>Proposed solution</u>: imitate the construction of algebraic spaces:

(1) start with a category of "local objects" = affine schemes

(2) Consider presheaves of sets on (aff sch) and the Yoneda embedding

$$\underline{\text{Yoneda}} : (\text{aff sch}) \hookrightarrow \begin{pmatrix}\text{presheaves} \\ \text{of sets} \\ (\text{aff sch})\end{pmatrix}$$

(3) Put a topology on (aff sch) e.g. the étale topology and get the obvious embedding

$$\begin{pmatrix}\text{Sheaves of} \\ \text{sets on} \\ (\text{aff sch})\end{pmatrix} \hookrightarrow \begin{pmatrix}\text{presheaves} \\ \text{of sets on} \\ (\text{aff sch})\end{pmatrix}$$

(4) <u>descent theory</u> show that the Yoneda embedding factors through

$$\begin{tikzcd}
(\text{aff sch}) \arrow[rr, "\text{Yoneda}"] \arrow[dr, hook] & & \begin{pmatrix}\text{presheaves} \\ \text{of sets}\end{pmatrix} \\
& \begin{pmatrix}\text{sheaves} \\ \text{of sets}\end{pmatrix} \arrow[ur] &
\end{tikzcd}$$

(5) Call a sheaf X - algebraic space if $\exists\ U_i \to X$ - étale s.t.

$$\coprod U_i \to X \text{ - surjective}$$

and all U_i - affine

<u>To mimick this with dg schemes</u>

(1) the category of local models is a 2-category.

<u>Fact</u>: (dga/k) is a simplicial model category

Objects: dga A, B, ...

<u>Morphisms</u>:

$$\mathrm{Hom}^{\Delta}(A, B)$$

simplicial Hom

is defined as follows.

Consider the cosimplicial scheme

$$\Delta_0 \rightrightarrows \Delta_1 \substack{\rightarrow\\\rightarrow\\\rightarrow} \Delta_2 \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

$$\Delta_n = \{ (x_0, \dots, x_n) \in \mathbb{A}^{n+1} \mid \sum x_i = 1 \}$$

$$\cdots \Omega(\Delta_n) \rightrightarrows \cdots \rightrightarrows \Omega(\Delta_1) \rightrightarrows \Omega(\Delta_0)$$

is a simplicial dga

Then $\mathrm{Hom}^\Delta(A, B) := \mathrm{Hom}_{dga}(A, B \otimes \Omega(\Delta))$

The model structure is defined as

<u>fibrations:</u> $A \twoheadrightarrow B$ - surjections
<u>weak eq.:</u> qis

<u>Remark:</u> if $A \rightarrowtail B$ - quasi free, then this is a cofibration.

<u>Corollary:</u> If A is quasi free $\Rightarrow$ A is both fibrant and cofibrant

Now if A - cofibrant and B - fibrant $\Rightarrow$

$\mathrm{Hom}^\Delta(A, B)$ is a Kan simplicial set (i.e. all horns complete to simplices)

Consider now the following 2-category R:

<u>objects</u>: quasi-free, f.g. dga concentrated in non-positive degrees

<u>Hom</u> $(A, B) = \Pi_1 \, \mathrm{Hom}^{\Delta}(A, B)$ fundamental groupoid

1-morphisms: morphism of dga $A \to B$

2-morphisms:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha \Downarrow"] & B \end{tikzcd}$$

is a homotopy class of maps.

<u>Def</u>: Take R^{opp} as the (2-category) of local models for dg schemes.

<u>Note</u>: The dg philosophy saying that dg schemes are smooth is already built into the definition here since our local models for dg geometry are chosen to be smooth

<u>Remark</u>: • In 2-categories one has two natural notions of fiber products:

$$\begin{tikzcd} \bullet \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Uparrow\alpha"] & & \\ & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ & \cdot \arrow[r] & \cdot \end{tikzcd}$$

weak : α exists

strong : α exists uniquely

• $\mathcal{R}^{opp}$ admits weak fiber products given by derived tensor products

$$(2) \quad \mathcal{R}^{opp} \overset{\text{2-Yoneda}}{\hookrightarrow} \left(\begin{array}{l} \text{presheaves 2-cat} / \mathcal{R}^{opp} \\ \text{fibers are grpds} \end{array} \right)$$

(3) étale topology on $\mathcal{R}^{opp}$

(4) <u>descent theory</u>

$$\mathcal{R}^{opp} \hookrightarrow (\text{sheaves})$$

(5) a sheaf X on $\mathcal{R}^{opp}$ is a dg scheme if it admits an affine covering.