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A d-categorical approach to dg schemes

dg schemes: "geometric objects" whose local coordinate rings are dg algebras

We will work only over k , $\text{char } k = 0$

A dg-algebra will mean a commutative unital dg-algebra over k .

Explicitly: • $A = \bigoplus_{i \geq 0} A^i$ graded algebra, $a \cdot b = (-1)^{\deg a \deg b} b \cdot a$

• A is equipped with a differential $d: A \rightarrow A$ of degree 1
($d^2 = 0$)

so that $d(ab) = (da)b + (-1)^{\deg a} a(db)$

Example: Let $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ be polynomials defining an affine scheme $V \subset \mathbb{A}^n$.

Consider the dga (A^\bullet, d) where

$$A^0 = k[x_0, \dots, x_n]$$

$$A^{-1} = A^0 \{f_1\} \oplus \dots \oplus A^0 \{f_r\}$$

$$A^{-p} = \bigoplus_{i_1 < \dots < i_p} A^0 \{f_{i_1} \dots f_{i_p}\}$$

i.e.

$$A^\bullet = \bigwedge_{A^0} A^1 \quad - \text{alternating algebra}$$

Define

$$d \sum_i f_i = \sum_i f_i \in A^0 \quad + \text{ extend using Leibniz}$$

Then (A^\bullet, d) is a dg algebra which is just the Koszul complex of $f_1, \dots, f_r \in A^0$.

Note:

- If we pass to cohomology \Rightarrow get a graded algebra $h^\bullet(A)$ over $h^0(A) = k[x_1, \dots, x_n] / (f_1, \dots, f_r)$
- f_1, \dots, f_r is a regular sequence $\Leftrightarrow \dim V = n-r \quad (\Rightarrow V \text{ l.c.i.})$
- $\Leftrightarrow A^\bullet \rightarrow h^\bullet(A)$ - resolution
- $\Leftrightarrow A^\bullet \rightarrow h^\bullet(A)$ - qis

Recall: A morphism $A \rightarrow B$ of dga is a quasi-isomorphism if

$$h^\bullet(A) \rightarrow h^\bullet(B)$$

is an isomorphism

Heuristically if $A \rightarrow h^\bullet(A)$ is a qis

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then we should think of $V = \text{Spec } k^0(A)$ as the dg-scheme defined by A .

Goal: generalise the above heuristics to general dga (not necessarily formal)

Note: The Koszul complex is an example of a quasi-free dga i.e.

$A^{\text{Koszul}} = (A \text{ without } d)$ is a graded commutative unital k -algebra

on the generators x_1, \dots, x_n (deg 0)
 ζ_1, \dots, ζ_r (deg -1)

Example: If we look at a standard affine open $\{g \neq 0\} \subset V$ for some $g(x) \in k[x_1, \dots, x_n]$, then we can adjoin two new variables

y (deg $y = 0$)
 ζ (deg $\zeta = -1$)

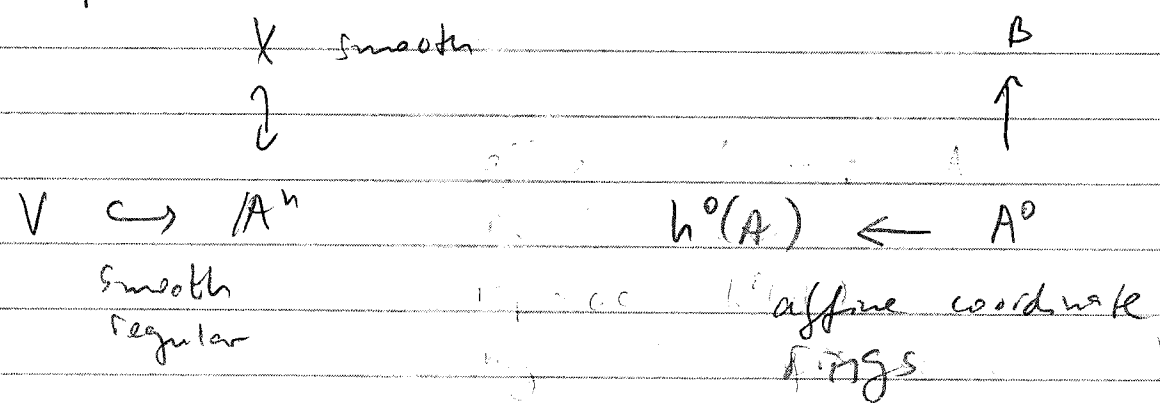
with the extra relation $d\zeta = y g^{-1}$
 then the resulting algebra

$k[x_1, \dots, x_n, y, \zeta_1, \dots, \zeta_r] / (d\zeta_i = f_i, d\zeta = y g^{-1})$

is again quasi free

In other words: localization along \mathfrak{g} in the dg world preserves freeness.

Example: (derived intersection)



Now in dg world replace $h^0(A)$ by a qis to it: A^\bullet .

Since A^\bullet is quasi free (\Rightarrow free as an A_0 module)

\Rightarrow

$$L^i(A^\bullet \otimes_{A_0} B) = \text{Tor}_{-i}^{A_0}(h^0(A), B)$$

dg-philosophy: fibered products of dg schemes should give derived tensor products

$$\left[\Rightarrow [V] \circ [X] = \text{td} \left(T_{A^n} - T_V - T_X \right)^{-1} \right. \\ \left. \cap \mathbb{E} \left(\text{For } \mathcal{O}_{A^n} (\mathcal{O}_V, \mathcal{O}_X) \right) \right]$$

Goal: Construct a category of dg schemes so that

- dg algebras define dg schemes
- every dg scheme has locally associated with it certain dgas, well defined up to qis
- fiber products of dg schemes are locally given by derived tensor products
- dg schemes can be glued together where qis are considered as isos

Example: Let U_1, U_2 two dg-schemes

Let

$$U_{1,g} \subset U_1 \quad - \text{open defined by } g \neq 0 \\ U_{2,h} \subset U_2 \quad - \text{open defined by } h \neq 0$$

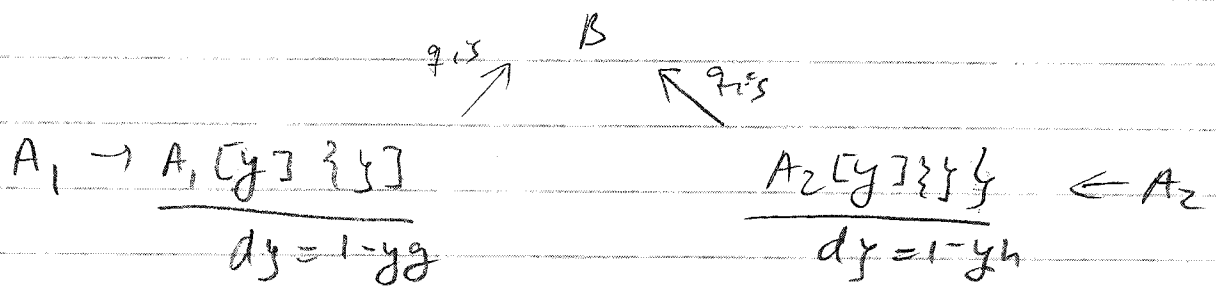
Assume that we are given q_{12}

$$U_{1, g} \xleftarrow{q_{12}} V \xrightarrow{q_{12}} U_{2, g}$$

then we should be able to glue U_1 and U_2 along V .

Explicitly if $U_1 \hookrightarrow \text{dga } A_1$
 $g \in A_1$
 $U_2 \hookrightarrow \text{dga } A_2$
 $h \in A_2$

then



should be a gluing diagram.

- derived moduli spaces should satisfy TMP in this category so that the dg structure on the moduli space should be determined by the TMP (universal mapping property).

Proposed solution: imitate the construction of algebraic spaces:

(1) start with a category of "local objects" = affine schemes

(2) Consider presheaves of sets on (aff sch) and the Yoneda embedding

$$\text{Yoneda} : (\text{aff sch}) \hookrightarrow \left(\begin{array}{l} \text{presheaves} \\ \text{of sets} \\ \text{on } (\text{aff sch}) \end{array} \right)$$

(3) Put a topology on (aff sch) e.g. the étale topology and get the obvious embedding

$$\left(\begin{array}{l} \text{Sheaves of} \\ \text{sets on} \\ \text{aff sch} \end{array} \right) \hookrightarrow \left(\begin{array}{l} \text{presheaves} \\ \text{of sets on} \\ \text{aff sch} \end{array} \right)$$

(4) descent theory show that the Yoneda embedding factors through

$$\begin{array}{ccc} (\text{aff sch}) & \xrightarrow{\text{Yoneda}} & \left(\begin{array}{l} \text{presheaves} \\ \text{of sets} \end{array} \right) \\ & \searrow & \nearrow \\ & \left(\begin{array}{l} \text{Sheaves} \\ \text{of sets} \end{array} \right) & \end{array}$$

(5) Call a sheaf X - algebraic space
if $\exists \mathcal{U}_i \rightarrow X$ - étale s.t.

$\coprod \mathcal{U}_i \rightarrow X$ - surjective
and all \mathcal{U}_i - affine

To mimic this with dg schemes

(1) the category of local models is
a 2-category.

Fact: (dga/k) is a simplicial model
category

Objects: $\text{dga } A, B, \dots$

Morphisms:

$\text{Hom}^\Delta(A, B)$

\uparrow
simplicial Hom

is defined as follows.

Consider the cosimplicial scheme

$$\Delta_0 \rightrightarrows \Delta_1 \rightrightarrows \Delta_2 \rightrightarrows \dots$$

$$\Delta_n = \{ (x_0, \dots, x_n) \in \mathbb{A}^{n+1} \mid \sum x_i = 1 \}$$

$$\mathcal{R}(\Delta_n) \xrightarrow{\sim} \mathcal{R}(\Delta_1) \xrightarrow{\sim} \mathcal{R}(\Delta_0)$$

is a simplicial dga

$$\text{Then } \text{Hom}^\Delta(A, B) := \text{Hom}_{\text{dga}}(A, B \otimes \mathcal{R}(\Delta))$$

The model structure is defined as

fibrations: $A \rightarrow B$ - surjectivity

weak eq.: qis

Remark: if $A \rightarrow B$ - quasi free, then this is a cofibration.

Corollary: If A is quasi free \rightarrow
 A is both fibrant and cofibrant

Now if A - cofibrant and B - fibrant

\rightarrow

$\text{Hom}^\Delta(A, B)$ is a Kan simplicial set (i.e. all horns complete to simplices)

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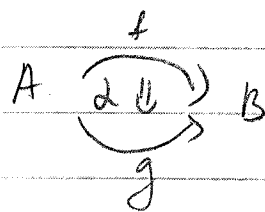
Consider now the following \mathcal{L} -category \mathcal{R} :

objects: quasi-free, f.g. dga
concentrated in non-positive
degrees

Hom $(A, B) = \prod_1 \text{Hom}^A(A, B)$ fundamental
groupoid

1-morphisms: morphism of dga $A \rightarrow B$

2-morphisms:

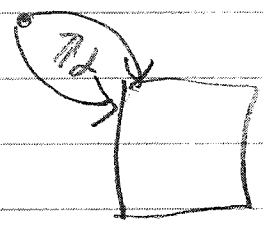


is a homotopy
class of
maps.

Def: Take \mathcal{R}^{opp} as the (\mathcal{L} -category)
of local models for dg
schemes.

Note: The dg philosophy saying that
dg schemes are smooth is
already built into the definition
here since our local models for
dg geometry are chosen to be
smooth

Remark: • In d -categories one has two natural notions of fiber products:



weak: d exists

strong: d exists uniquely

• \mathcal{R}^{opp} admits weak fiber products given by derived tensor products

(2) $\mathcal{R}^{opp} \xrightarrow{2\text{-Yoneda}}$ (presheaves 2-cat / \mathcal{R}^{opp})
fibers are gpd's

(3) étale topology on \mathcal{R}^{opp}

(4) descent theory

$\mathcal{R}^{opp} \hookrightarrow$ (sheaves)

(5) a sheaf X on \mathcal{R}^{opp} is a dg algebra if it admits an affine covering.