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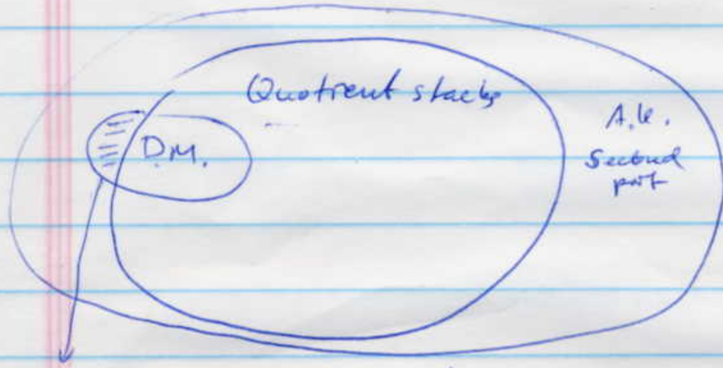
Intersection theory on algebraic stacks: The approach of Totaro, Edidin and Graham

- Continuation of A. Kresch's lecture from the first part of the workshop - DM-stacks



we do not know what are all of the complement

Recall

Quotient stack (over a field k)

G - linear group (affine) /k.

G - acts on X

$[X/G]$ q. stack

~~the~~ category, fibered over Sch/k

$$\begin{array}{c}[X/G] \\ \downarrow \\ Sch/k\end{array}$$

$$ob\,[X/G] = \left\{ \begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \\ T & & \end{array} \;\middle|\; \begin{array}{l} P \\ \downarrow \\ T \end{array} \text{ is } G \text{ torsor (principal } G\text{-bundle) } \& \;\; P \longrightarrow X \text{ is } G \text{ equiv. map} \right\}$$

Arrows: in the obvious way.

The functor : $[X/G] \longrightarrow Sch/k$

is defined :

$$\begin{pmatrix} P \longrightarrow X \\ \downarrow \\ T \end{pmatrix} \longmapsto T$$

E.g. • BG — classifying stack of G

$ob(BG) =$ principal G-bundles

$BG = [\operatorname{spec} k / G]$

• $\overline{\mathcal{M}}_g$ — is a quotient stack.

How to prove that a stack X is a quotient stack :

• rigidify the objects of X

for $X = \overline{\mathcal{M}}_g$ take $n \gg 0$

• Consider $\mathcal{E}_g = \left\{ \left(\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix}, \varphi \right) \;\middle|\; \begin{smallmatrix} C \\ \pi \downarrow \\ S \end{smallmatrix} \in ob\, \overline{\mathcal{M}}_g \right.$ - stable curve

$\varphi : \mathcal{O}_S^N \xrightarrow{\sim} \pi_* \omega_{C/S}^{\otimes n}$

• and a map

$$\begin{array}{ccc} \mathcal{E}_g & & \left(\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix}, \varphi \right) \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_g & & \left(\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix} \right) \end{array}$$

• Note : $\mathcal{E}_g$ is a principal GL_n bundles over $\overline{\mathcal{M}}_g$

($\mathcal{E}_g$ - projective bundle associated with the vector bundle $\pi_* \omega_{C/S}^{\otimes n}$.)

in fact, $E_g = \mathcal{E}_g$ is a quasi proj variety $/k$

and,

$$[E_g / GL_N] \xrightarrow{\simeq} \overline{M_g}$$

pf. : Define $\overline{M_g} \longrightarrow [E_g/GL_N]$

$$\begin{array}{c} C \\ \downarrow \pi \\ S \end{array}$$

has associated v.b. $\pi_* \omega_{C/S}^{\otimes n}$

$\updownarrow$

$\leadsto$ principal GL_N bundle of frames over $\pi_* \omega_{C/S}^{\otimes n}$.

$$\begin{array}{c} P \\ \downarrow \\ S \end{array}$$

Pull-back

$$\begin{tikzcd} P \times_S C \arrow[r, "g"] \arrow[d, "\rho"'] & P \arrow[d, "\pi"] \\ C \arrow[r, "f"] & S \end{tikzcd}$$

$$\rho_* \omega_{P \times_S C} \simeq \mathcal{O}_P^N$$

gives you a map

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E_g \\ S & \end{tikzcd}$$

The opposite map $[E_g/GL_N] \longrightarrow M_g$ comes from descent theory.

Suppose $\mathcal{X} = [X/G]$

Want to define $A_* X$ – Chow

<u>Idea</u>: Borel construction.

In topology: replace non-free action of G on X by a free action:
There exists a contractible space E on which G acts freely and take $X_G = X \times E/G$

Then $H^*_G(X) = H^*(X_G)$

Idea of Totaro:

We cannot exactly construct the contractible E but we can approximate it in any codimension

For $m > 0$, $\exists\ G \to GL(V)$ $U \subset V$ open ~~subspace~~ subscheme
G-invariant G acts on U freely,
$\operatorname{codim}_V (V \setminus U) > m$

We take $X_G = X \times U/G$

<u>Definition</u> $A^G_i X = A_{i + \dim V - \dim G}(X \times U/G)$ for $i \leq \dim X - m$
(i.e., up to codim $= m$)

$X \times U/G$ is alg space — it has a Chow ring

Inductively this defines $A_i \mathcal{X}$.

(Need to prove correctness – independent of the quotient ~~stack~~ structure on the stack.

Theorem This is well defined ($A_*^G \mathcal{X}$)

$A_* \mathcal{X}$ is contravariant for flat and l.c.i maps

Has a ring structure when $\mathcal{X}$ is smooth
A_* is a covar. functor (proper repres. maps)

Homotopy: $\mathcal{V} \xrightarrow{\pi} \mathcal{X}$ vector bundle
$\pi^* A_* \mathcal{X} \xrightarrow{\sim} A_* \mathcal{V}$ is an iso

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$\exists$ Chern classes

If $\mathcal{X}$ smooth

Thm.: $c \quad Pic(\mathcal{X}) \longrightarrow A^1(\mathcal{X}) = A_{\dim \mathcal{X}-1}(\mathcal{X})$
is an isomorphism

Thm: If $\mathcal{X}$ is D-M stack M - moduli space
M is locally sep. alg space
$M \xrightarrow{\Delta} M \times M$ is embedding, then

$A_*(\mathcal{X}) \otimes \mathbb{Q} \cong A_* M \otimes \mathbb{Q}$

(The new intersection theory on DM stacks agrees with the 'old' ~~too~~ intersection theory)

Examples BGL_n

Consider $X_s = \{$ all n-independent vectors in $k^{n+s}\}$

$$X_s \subseteq (k^{n+s})^n$$

$$\operatorname{codim} (k^{n+s})^n \setminus X_s \longrightarrow \infty \quad \text{when } s \to \infty$$

The quotient $X_s / GL_n = G(n, n+s)$ Grassmannian

$$A^*(G(n,n+s)) = \mathbb{Z}[c_1, \dots c_n] / (\text{relations in higher degrees})$$

$\longrightarrow \infty \rightsquigarrow$ relations are lost

upshot: $A^* BGL_n = \mathbb{Z}[c_1, \dots c_n]$

Exercise. $A^* B\mu_n \qquad \mu_n = \{\sqrt[n]{1}\}$

μ_n acts on $k^s \setminus (0)$
$\to$ total space of $\mathcal{O}_{\mathbb{P}k^s}(n)$

~~B~~ Answer: $A^* B\mu_n = \mathbb{Z}[t] / nt$

Note: $A^i \mathcal{X}$ is non-zero for $i > \dim \mathcal{X}$
$\rightsquigarrow$ nonzero in negative dimension

Example $\mathcal{M}_{1,1}$, $\overline{\mathcal{M}_{1,1}}$

$$M_{1,1} = \mathbb{A}^1, \qquad \overline{M_{1,1}} = \mathbb{P}^1$$

the Chow rings of the stacks are different:

(Mumford) $\rightarrow A^1 \mathcal{M}_{11} \simeq \operatorname{Pic} \mathcal{M}_{1,1} = \mathbb{Z}/12\mathbb{Z}$

Theorem
$$A^* \mathcal{M}_{1,1} = \mathbb{Z}[t] / (12t)$$

$$A^* \overline{\mathcal{M}_{1,1}} = \mathbb{Z}[t] / 24t^2$$

$$(\lambda_1 = t = c_1(\mathbb{E}))$$

Proof: We write $\mathcal{M}_{11}$ and $\overline{\mathcal{M}_{11}}$ as quotient stacks;

Canonical form of ~~an~~ elliptic curve:

$C_{a,b} : y^2 = x^3 + ax + b$

canonical diff. $0 \neq \frac{dx}{y} \in H^0(\omega_{C_{a,b}})$

$$\mathbb{A}^2 \supseteq X_g = \{(a,b) \mid \Delta \neq 0\}$$

$$\Delta = -4a^3 - 27b^2 \neq 0$$

$$\overline{X_g} = \{(a,b) \mid \text{no triple roots i.e., } (a,b) \neq (0,0)\}$$

Canonical family of elliptic curves
$$C \downarrow X_1 \qquad \overline{C} \downarrow \overline{X_1}$$

Exercise: X_1, $\overline{X_1}$ are the total spaces of the Hodge bundles;

on $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$

(to assign a differential on an elliptic curve $\leftrightarrow$ choosing a repres. in these families)

$\mathbb{G}_m$ acts on $X_{1,1}$ $\overline{X}_{1,1}$

$$t(x,y) = (t^{-2}x, t^{-3}y)$$

note: $\dfrac{d(t^{-2}x)}{t^{-3}y} = t\dfrac{dx}{dy}$ — incorrect action on the bundle of frames of the Hodge bundle.

$$t(y^2 = x^3 + ax + b) \Longrightarrow (y^2 + (t^4 a)x + t^6 b)$$
$$t : (a, b) \longmapsto (t^4 a, t^6 b)$$

$\rightsquigarrow$ Exact sequence:

$$\begin{tikzcd}
 & A^*_{\mathbb{G}_m}(X_{1,1}) \arrow[d, phantom, "\shortparallel"] & \\
A^*_{\mathbb{G}_m}(0) \arrow[r] \arrow[d, phantom, "\shortparallel"] & A^*_{\mathbb{G}_m}(\mathbb{A}^2) \arrow[r] \arrow[d, phantom, "\shortparallel"] & A^*\overline{\mathcal{M}}_{1,1} \arrow[r] & 0 \\
A^*(B\mathbb{G}_m) \arrow[r] \arrow[d, phantom, "\shortparallel"] & A^*_{\mathbb{G}_m} & \\
\mathbb{Z}[t] \arrow[r] & \mathbb{Z}[t] &
\end{tikzcd}$$

$\uparrow$ multiplication by $c_2(\mathbb{A}^2_{(a,b)}) = \dots = 4t \cdot 6t = 24t^2$

Example $\mathcal{M}_2$ - much less trivial

- X_2 - total space of Hodge GL_2 bundle
- X_2 open $\subseteq \mathbb{A}^7 = \{$ sextic forms $\}$

$y^2 = f(x_0, x_1)$

To write a $g=2$ curve in this form is equiv:

$$X_2 = \{ f \in \mathbb{A}^7 \mid \Delta f \neq 0 \}$$

Action of GL_2 on X_2

$$(Af)(x) = (\det A)^2 f(A^{-1}x)$$

$$\lambda_i = c_i(\mathbb{E}_2)$$

$$A^* \mathcal{M}_2 = \mathbb{Z}[\lambda_1, \lambda_2] / (10\lambda_1, 2\lambda_1^2 - 24\lambda_2)$$

Question : Do you have the result for $\mathcal{M}_{g,n}$

Ans : very hard.

Q : Hyper elliptic curves

Q :