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Can one do arithmetic intersection theory on stacks?

Plan:
- Review of arithmetic intersection theory
- sketch the plan for DM stacks
- ? Artin stacks.

Arithmetic intersection theory

Look at schemes $/\mathbb{Z}$ - projective and flat over $\mathbb{Z}$

We would like to be able to do intersection theory on such a $\mathfrak{X} \to \operatorname{Spec}\mathbb{Z}$ in a way that will produce numbers.

Analogy: Take a field k, $U = C - \{\infty\}$ where C - projective curve $/k$
$\infty \in C$.

Look at

$$\mathfrak{X} \to U$$

scheme projective and flat $/U$.

If we want to get intersection numbers on $\mathfrak{X}$ we run into trouble since $\mathfrak{X}$ is not projective $/k$

To remedy this one looks at a compactification

$$\overline{\mathcal{X}} \to C \quad \text{of} \quad \mathcal{X} \to C$$

s.t. $\overline{\mathcal{X}}$ - non-singular + projective over k

and do intersection theory on $\overline{\mathcal{X}}$.

Thus we may look at

$$\varinjlim_{\text{all } \mathcal{X} \subset \overline{\mathcal{X}}} CH^\bullet(\overline{\mathcal{X}}) =: \widehat{CH}^\bullet(\mathcal{X}).$$

The key remark is that $\widehat{CH}^\bullet(\mathcal{X})$ has intersection numbers i.e. the natural maps

$$\begin{tikzcd} CH^p(\overline{\mathcal{X}}) \arrow[d, shift right] \arrow[dr, "\deg"] & \\ CH^p(\overline{\mathcal{X}}') \arrow[u, shift right] \arrow[r, "\deg"'] & \mathbb{Z} \end{tikzcd}$$

commute.

<u>Example</u>: For $p=1$ we have

$$\widehat{CH}^1(\mathcal{X}) \cong \widehat{Pic}(\mathcal{X}) = \text{iso classes of line bundles on } \mathcal{X} + \text{extensions of these line bundles to some } \overline{\mathcal{X}}$$

It turns out that this is the same as

$$\left(\begin{array}{l}\text{Isomorphism classes of line} \\ \text{bundles on } \mathfrak{X} \text{ equipped} \\ \text{with a } v\text{-adic metric}\end{array}\right)$$

Here v is the valuation corresponding to $\infty \in C$.

What should one do in the arithmetic situation?

Compactify $\operatorname{Spec} \mathbb{Z}$ as

$$\overline{\operatorname{Spec} \mathbb{Z}} = \operatorname{Spec} \mathbb{Z} \cup \{\infty\}$$

where $\infty : \mathbb{Q} \hookrightarrow \mathbb{R}$ is the archimedean valuation of $\mathbb{Q}$.

<u>Now we</u> can define a bundle on $\overline{\operatorname{Spec} (\mathbb{Z})}$ as the data of:

- a bundle on $\operatorname{Spec} \mathbb{Z}$ i.e. a free $\mathbb{Z}$-module $\mathcal{L}$
- inner product h on $\mathcal{L} \otimes \mathbb{R}$

This definition is consistent with the adelic point of view + we can define degree:

If $(\mathcal{L}, h)$ is a vector bundle on $\mathrm{Spec}(\mathbb{Z})$ $\Rightarrow$ define

$$\deg(\mathcal{L}, h) = -\log\begin{pmatrix}\text{covolume of} \\ \mathcal{L} \subset \mathcal{L}\otimes\mathbb{R}\end{pmatrix}$$

More generally given a scheme

$$\mathfrak{X} \to \mathrm{Spec}\,\mathbb{Z} \qquad \begin{pmatrix}\text{projective} \\ + \text{ flat}\end{pmatrix}$$

we may look at compactifications

$$\begin{tikzcd} \mathfrak{X} \arrow[d] & \cup & \mathfrak{X}(\mathbb{C}) \arrow[d] \\ \mathrm{Spec}\,\mathbb{Z} & \cup & \infty \end{tikzcd}$$

$$\infty = \mathbb{Q} \hookrightarrow \mathbb{R}$$

Then we can define (similarly to the geometric case): $\widehat{\mathrm{Pic}}(\mathfrak{X}) =$

iso classes of pairs $(\mathcal{L}, h)$
s.t. $\mathcal{L}$ = line bundle on $\mathcal{X}$
h = hermitian inner product on $\mathcal{L}|_{\mathcal{X}(\mathbb{C})}$

<u>Parshin-Arakelov</u>: There is an intrinsically defined intersection pairing

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathbb{R}$$

<u>Example</u>: On $\mathbb{P}^n_{\mathbb{Z}}$ we have

$$\mathcal{O}^{n+1} \twoheadrightarrow \mathcal{O}(1)$$

and we can extend $\mathcal{O}(1)$ to $\overline{\mathcal{O}(1)}$ on $\mathbb{P}^n_{\mathbb{Z}}$ and so for every point

$$x \in \mathbb{P}^n(\mathbb{Q})$$

we have the corresponding

$$\overline{x} : \operatorname{Spec} \mathbb{Z} \to \mathbb{P}^n_{\mathbb{Z}}$$

and it turns out that

$$\widehat{\deg}(\overline{x}^* \overline{\mathcal{O}(1)}) = \mathrm{height}(x).$$

In general: given a regular flat projective

$$X \longrightarrow \operatorname{Spec} \mathbb{Z}$$

we can again define

$$\widehat{CH}^q(X) = \text{codim } q \text{ arithmetic cycles on } X.$$

To do this reexamine the case of $\widehat{\mathrm{Pic}}(X)$: Given $(\mathcal{L}, h) \in \widehat{\mathrm{Pic}}(X)$ we can consider $s \in \mathcal{L}$ and attach to it the pair

$$\left(\mathrm{div}(s), -\log \|s\|_h^2\right)$$

<u>Note</u>: this satisfies

$$dd^c(-\log\|s\|) + \delta_{\mathrm{div}(s)} = c_1(\mathcal{L})$$

For higher codimension we put

$\widehat{CH}^q(X) =$ pairs (Z, g) where

- Z is a codimension q cycle on X
- g is a current on $X(\mathbb{C}) - Z(\mathbb{C})$ satisfying

$$dd^c g + \delta_Z = \omega \leftarrow C^\infty \ (q,q)\text{-form on } X(\mathbb{C})$$

We can define intersection for such pairs (Z, g_Z), (Y, g_Y):

$$(Z, g_Z)\cdot(Y, g_Y) := (Z\cdot Y,\ \delta_Y g_Z + g_Y \omega_Z)$$

The case of DM stacks

Let now $\mathcal{X}$ be a DM stack over $\operatorname{Spec}\mathbb{Z}$.

We will only consider $\mathbb{Q}$-coefficients for the groups of cycles

Choose an étale groupoid presentation

$$\mathcal{X} \leftarrow U_0 \leftleftarrows U_1$$

and work with the corresponding simplicial scheme

$$\mathcal{U}_0 \leftleftarrows \mathcal{U}_1 \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; \mathcal{U}_1 \times_{\mathcal{U}_0} \mathcal{U}_1 \;\cdots \;\text{---}$$

Now define a group of cycles:

$$Z^p(\mathfrak{X})_{\mathbb{Q}} := H^0\big(Z^p(\mathcal{U}_0) \rightrightarrows Z^p(\mathcal{U}_1) \to \cdots \big)$$

= rational vector space spanned by the integral substances of codimension p on $\mathfrak{X}$.

Define a group of rational equivalences:

$$R^{p-1}(\mathfrak{X}) = H^0\big(R^{p-1}(\mathcal{U}_0) \rightrightarrows R^{p-1}(\mathcal{U}_1) \cdots \big)$$

$$\| $$

$$\bigoplus_{\substack{\text{point } z \text{ of}\\ \text{codimension } p-1}} k(z)^*$$

$$= \bigoplus_{\substack{Z \subset \mathfrak{X}\\ \text{integral codim } p-1}} k(Z)^*$$

We still have a divisor class map

$$\mathrm{div} : R^{p-1}(\mathcal{X}) \to Z^p(\mathcal{X})_{\mathbb{Q}}$$

and we can form the quotient.

What to do with the currents?

<u>Burgos</u>: If X - complex projective manifold $\Rightarrow$ the Green current "g_Y" of $Y \subset X$ can be rewritten in terms of differential forms with logarithmic growth.

$$X \subset \overline{X} \qquad , \qquad \overline{X} - X = \bigcup D_i = D$$

projective $\qquad\qquad$ Normal crossings

Consider $\mathcal{E}^{\bullet}_{\log}(\overline{X}, D)$ - forms involving

$$\log |z_i|^2 , \ \frac{dz_i}{z_i} , \ \frac{d\overline{z_i}}{\overline{z_i}}$$

Then Burgos showed that the MHS on X can be described in terms of $\mathcal{E}^{\bullet}_{\log}(\overline{X}, D)$ and that moreover for any $Y \subset X \Rightarrow$ "g_Y" can be in terms of forms like that on $X - Y$.

In fact with Burgos' interpretation of Green currents on schemes we get homotopy invariance of the Arithmetic intersection theory:

For a vector bundle $E \to X$ we have

$$\widehat{CH}^p(X) \cong \widehat{CH}^p_{ver}(E)$$

$$\shortparallel$$

(Y, g_Y) s.t.

$$dd^c g_Y + \delta_Y = \omega$$

and ω is pulled back from X

In particular we can use Burgos currents to define $g_{\mathcal{Y}}$ for a substack

$$\mathcal{Y} \subset \mathcal{X}$$

of codimension q.

Namely look at

$$\begin{array}{ccccccc}
\mathcal{X} & \leftarrow & U_0 & \leftleftarrows & U_1 & \Lleftarrow & \cdots \\
\cup & & \cup & & \cup & & \\
\mathcal{Y} & \leftarrow & Y_0 & \leftleftarrows & Y_1 & & \cdots
\end{array}$$

and define g_Y in terms of $\mathcal{E}^*_{log}(U-Y)$ by choosing a normal crossing compactification of $U-Y$.

The next question to address is the existence of specialization maps – what happens to the currents when we deform to the normal cone: use asymptotic expansion of the current in the t-parameter (the parameter for the deformation to the normal cone).

It follows from the 1979 thesis of J. Hu that if $Z \subset X$ is a cycle on a smooth variety $/\mathbb{C}$, and $Y \hookrightarrow X$ is a closed immersion of smooth varieties that, given a Green form g_Z for Z, taking the finite part of this expansion gives a Green current (not necessarily a form!) for the cone $C_{Y\cap Z}Z \subset N_{X/Y}$.

It still remains to check that this pull back defines a product which is commutative - this is work in progress.

Artin Stacks

What can one say about Artin stacks? Since such stacks may not be proper over $\operatorname{Spec} \mathbb{Z}$ (e.g. BGL_n is not separated over $\operatorname{Spec} \mathbb{Z}$) it is not clear what one might mean by a "compactification" over the real embedding $\mathbb{Q} \subset \mathbb{R}$

Note in particular that $BGL_n \mathbb{Z}$ has no interesting Hermitian vector bundles, since these would correspond to unitary representations of $GL_n \mathbb{Z}$.