

Henri Gillet

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Can one do arithmetic intersection theory on stacks?

- Plan:
- Review of arithmetic intersection theory
 - sketch the plan for JM stacks
 - ? Artin stacks.

Arithmetic intersection theory

Look at schemes $/\mathbb{Z}$ - projective and flat over \mathbb{Z}

We would like to be able to do intersection theory on such a $\mathcal{X} \rightarrow \text{Spec } \mathbb{Z}$ in a way that will produce numbers.

Analogy: Take a field k , $U = C - \{\infty\}$ where C - projective curve $/k$
 $\infty \in C$.

Look at

$\mathcal{X} \rightarrow U$

scheme projective and flat $/U$.

If we want to get intersection numbers on \mathcal{X} we run into trouble since \mathcal{X} is not projective $/k$.

To remedy this one looks at a compactification

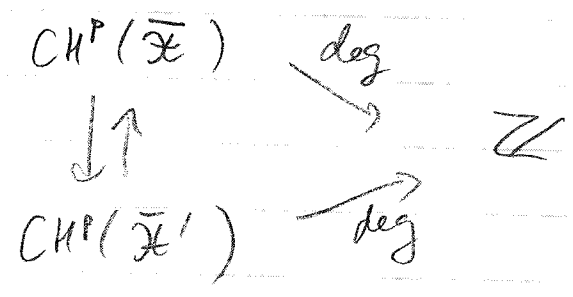
s.t. $\bar{X} \rightarrow C$ of $X \rightarrow C$
non-singular & projective over k

and do intersection theory on \bar{X} .

Thus we may look at

$$\lim_{\substack{\longrightarrow \\ \text{all } \bar{X}' \subset \bar{X}}} CH^*(\bar{X}') =: \hat{CH}^*(X)$$

The key remark is that $\hat{CH}^*(X)$ has intersection numbers i.e. the natural maps



commute.

Example: for $p=1$ we have

$\hat{CH}^1(X) \cong \text{Pic}^0(X) =$ iso classes
of line bundles
on X + extensions
of these line bundles
to some \bar{X}

It turns out that this is the same as

(Isomorphism classes of line bundles on \mathbb{A}^1 equipped with a v -adic metric)

Here v is the valuation corresponding to $\infty \in \mathbb{C}$.

What should one do in the arithmetic situation?

Compactify $\text{Spec } \mathbb{Z}$ as

$$\overline{\text{Spec } \mathbb{Z}} = \text{Spec } \mathbb{Z} \cup \{\infty\}$$

where $\varphi: \mathbb{Q} \hookrightarrow \mathbb{R}$ is the archimedean valuation of \mathbb{Q} .

Now we can define a bundle on $\text{Spec } (\mathbb{Z})$ as the data of:

- a bundle on $\text{Spec } \mathbb{Z}$ i.e. a free \mathbb{Z} -module \mathbb{I}
- inner product h on $\mathbb{I} \otimes \mathbb{R}$

This definition is consistent with the adelic point of view + we can define degree:

If (\mathcal{L}, h) is a vector bundle on $\text{Spec}(\mathbb{Z}) \Rightarrow$ define

$$\deg(\mathcal{L}, h) = -\log\left(\frac{\text{covolume of } \mathcal{L} \subset \mathcal{L} \otimes \mathbb{R}}{\text{vol}(\mathbb{R})}\right)$$

More generally given a scheme

$$\mathcal{X} \rightarrow \text{Spec } \mathbb{Z} \quad \left(\begin{array}{l} \text{projective} \\ + \text{flat} \end{array} \right)$$

we may look at compactifications

$$\begin{array}{ccc} \mathcal{X} & \cup & \mathcal{X}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \text{Spec } \mathbb{Z} & \cup & \infty \\ & & \text{"} \\ & & \mathbb{Q} \hookrightarrow \mathbb{R} \end{array}$$

Then we can define (similarly to the geometric case): $\widehat{\text{Pic}}(\mathcal{X}) =$

iso classes of pairs (\mathcal{L}, h)
 s.t. $\mathcal{L} =$ line bundle on X
 $h =$ hermitian inner product on
 $\mathcal{L}|_{X(\mathbb{C})}$

Parshin-Aravelov: There is an intrinsically
 defined intersection pairing

$$\widehat{\text{Pic}}(X) \times \widehat{\text{Pic}}(X) \longrightarrow \mathbb{R}$$

Example: On $\mathbb{P}^n_{\mathbb{Z}}$ we have

$$\mathcal{O}^{h+1} \longrightarrow \mathcal{O}(1)$$

and we can extend $\mathcal{O}(1)$ to $\overline{\mathcal{O}(1)}$
 on $\mathbb{P}^n_{\mathbb{Z}}$ and so for every point

$$x \in \mathbb{P}^n(\mathbb{Q})$$

we have the corresponding

$$\bar{x}: \text{Spec } \mathbb{Z} \rightarrow \mathbb{P}^n_{\mathbb{Z}}$$

and it turns out that

$$\widehat{\text{deg}}(\bar{x}^* \overline{\mathcal{O}(1)}) = \text{height}(x).$$

In general: given a regular
flat projective

$$X \rightarrow \text{Spec } \mathbb{Z}$$

we can again define

$$\widehat{CH}^q(X) = \text{codim } q \text{ arithmetic cycles on } X.$$

To do this re-examine the case of $\widehat{\text{Pic}}(X)$: Given $(\mathcal{L}, h) \in \widehat{\text{Pic}}(X)$ we can consider $s \in \mathcal{L}$ and attach to it the pair

$$\left(\text{div}(s), -\log \|s\|_h^2 \right)$$

Note: this satisfies

$$dd^c(-\log \|s\|) + \delta_{\text{div}(s)} = c_1(\mathcal{L})$$

For higher codimension we put

$$\widehat{CH}^q(X) = \text{pairs } (Z, g) \text{ where}$$

- Z is a codimension g cycle on X
- g is a current on $X(\mathbb{C}) - Z(\mathbb{C})$ satisfying

$$dd^c g + \delta_Z = \omega \leftarrow \begin{array}{l} \text{a } (g, g)\text{-form} \\ \text{on } X(\mathbb{C}) \end{array}$$

We can define intersection for such pairs $(Z, g_Z) \rightarrow (Y, g_Y) :$

$$(Z, g_Z) \cdot (Y, g_Y) := (Z \cdot Y, \delta_Y g_Z + g_Y \omega_Z)$$

The case of DM stacks

Let now \mathcal{X} be a DM stack over $\text{Spec } \mathbb{Z}$.

We will only consider \mathbb{Q} -coefficients for the groups of cycles.

Choose an étale groupoid presentation

$$\mathcal{X} \leftarrow \mathcal{U}_0 \rightrightarrows \mathcal{U}_1$$

and work with the corresponding
simplicial scheme

$$U_0 \leftarrow U_1 \xrightarrow{\cong} U_1 \times_{U_0} U_1 \xrightarrow{\cong} \dots$$

Now define a group of cycles:

$$Z^p(\mathcal{X})_{\mathbb{Q}} := H^0(Z^p(U_0) \rightrightarrows Z^p(U_1) \rightarrow \dots)$$

= rational vector space
spanned by the
integral substances of
codimension p on \mathcal{X} .

Define a group of rational equivalences:

$$R^{p-1}(\mathcal{X}) = H^0(R^{p-1}(U_0) \rightrightarrows R^{p-1}(U_1) \rightarrow \dots)$$

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$$\bigoplus k(z)^*$$

point z of
codimension $p-1$

$$= \bigoplus_{z \in \mathcal{X}} k(z)^*$$

$z \in \mathcal{X}$

integral codim $p-1$

We still have a divisor class map

$$\text{div} : \mathbb{R}^{P-1}(X) \rightarrow \mathbb{Z}^P(X)_{\mathbb{Q}}$$

and we can form the quotient.

What to do with the currents?

Burgos: If X - complex projective manifold \Rightarrow the Green current " g_Y " of $Y \subset X$ can be rewritten in terms of differential forms with logarithmic growth.

$$X \subset \bar{X}, \quad \bar{X} = X = \cup D_i = \mathcal{D}$$

projective Normal crossings

Consider $\mathcal{E}_{\log}^*(\bar{X}, \mathcal{D})$ - forms involving

$$\log |z_i|^2, \frac{dz_i}{z_i}, \frac{d\bar{z}_i}{z_i}$$

Then Burgos showed that the MHS on X can be described in terms of $\mathcal{E}_{\log}^*(\bar{X}, \mathcal{D})$ and that moreover for any $Y \subset X \Rightarrow$ " g_Y " can be

In terms of forms like that on $X - Y$.

In fact with Burgos' interpretation of Green currents on schemes we get homotopy invariance of the Arithmetic intersection theory:

For a vector bundle $E \rightarrow X$ we have

$$\widehat{CH}^p(X) \cong \widehat{CH}_{\text{vec}}^p(E)$$

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(Y, g_Y) s.t.

$$dd^c g_Y + \delta_Y = \omega$$

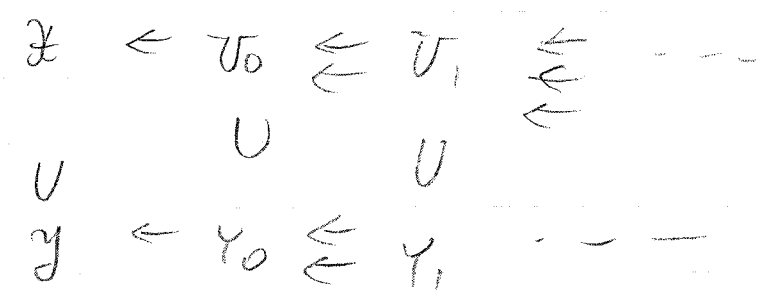
and ω is pulled back from X

In particular we can use Burgos currents to define g_Y for a subvariety

$$Y \subset X$$

of codimension q .

Namely look at



and define g_Y in terms of $\sum^* \log(U-Y)$ by choosing a normal crossing compactification of $U-Y$.

The next question to address is the existence of specialization maps - what happens to the currents when we deform to the normal cone: use asymptotic expansion of the current in the t -parameter (the parameter for the deformation to the normal cone).

It follows from the 1999 thesis of J. Hu that if $Z \subset X$ is a cycle on a smooth variety $/\mathbb{C}$, and $Y \hookrightarrow X$ is a closed immersion of smooth varieties that, given a Green form ω_Z for Z , taking the finite part of this expansion gives a Green current (not necessarily a form!) for the cone $C_{Y,Z} \subset N_{X/Y}$.

It still remains to check that this pull back defines a product which is commutative - this is work in progress.

Artin Stacks

What can one say about Artin stacks? Since such stacks may not be proper over $\text{Spec } \mathbb{Z}$ (e.g. BGL_n is not separated over $\text{Spec } \mathbb{Z}$) it is not clear what one might mean by a "compactification" over the real embedding $\mathbb{Q} \subset \mathbb{R}$.

Note in particular that $BGL_n \mathbb{Z}$ has no interesting Hermitian vector bundles, since these would correspond to unitary representations of $GL_n \mathbb{Z}$.