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Intersection theory of algebraic stacks - Chow groups for Artin stacks

We will consider stacks of finite type over base field k which have the property that the stabilizer group of every geometric point is a linear algebraic group schemes (possible non-reduced

Outline of IT for DM stacks

- definition of $A_\circ X = \dfrac{\text{algebraic cycles}}{\text{rational equivalence}}$
- construction of the basic operations (follows from the construction for schemes and étale base change)
  - flat pullback
  - intersection with a principal Cartier divisor
  - proper pushforward
    - ok for proper representable maps
    - in general works only $\otimes \mathbb{Q}$.
- Homotopy invariance for vector bundles (works $\otimes \mathbb{Q}$).

- All of the above allow one to build intersection theory via deformation to the normal cone

- (Punchline) If X - smooth DM stack, then the diagonal $X \longrightarrow X \times X$ is a representable l.c.i unramified morphism $\Rightarrow$ get an intersection product.

The case of Artin stacks

We will follow the above outline from the bottom up.

- (Punchline) A smooth Artin stack X has representable l.c.i diagonal but the diagonal map has usually big fibers.

As a consequence the tangent stack TX of an Artin stack is not a vector bundle on X but is a vector bundle stack i.e. is locally a quotient of one vector bundle by the additive action of another one. This new feature of Artin stacks requires us to modify the other steps of building IT accordingly

- We can only hope for deformation to the normal cone stack. In fact the usual construction works: Given a representable l.c.i

$$Y \hookrightarrow X$$

of Artin stacks we can form

$$M^{0}_{X/Y} \to \mathbb{P}^1$$

s.t. : general fiber $\cong Y$
special fiber $\cong C_{X/Y}$ - the normal cone stack of $Y \subset X$.

- Homotopy invariance for vector bundle stacks

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & {[F/E]} \arrow[d] \\ & X \end{tikzcd}$$

works verbatim

and one needs to do more work for v.b. stacks which are not necessarily global quotients.

- For the basic operations we will need some extra info – postponed for now.

- For the definition of $A_\bullet \mathfrak{X}$ – reexamine the case of quotient stacks first.

Recall (from Vistoli's lecture yesterday):

If $\mathfrak{X} = [X/G]$, then the Totaro – Edidin – Graham formalism gives

$$A_i \mathfrak{X} := A_i^G X = A^0_{i+\text{shift}}\left(\frac{X \times U}{G}\right)$$

$$\| $$

$$A^0_{i+\text{shift}}\left([X \times V/G]\right)$$

$\dfrac{\text{Cycles}}{\text{rat eq.}}$

vector bundle on $\mathfrak{X}$

(Here $U \subset V$ is an equivariant open on which G acts freely)

This picture motivates the following

Def: For an Artin stack $\mathcal{X}$ define

$$\hat{A}_i \mathcal{X} = \varinjlim_{\substack{\text{all v.b.} \\ \mathcal{E} \to \mathcal{X}}} A^0_{i+\operatorname{rk}\mathcal{E}}(\mathcal{E})$$

Remark: When $\mathcal{X}$ is a quotient stack, one gets this way the Chow groups from yesterday's lecture.

Remark: The limit above is taken along all vector bundles and surjective maps of such.

Unfortunately the groups $\hat{A}_i$ are not covariantly functorial

Example: Let $\mathcal{X}$ be a $\mathbb{Z}/2$ gerbe over a scheme T. Suppose that $\mathcal{X}$ has no nontrivial vector bundles (there are examples like that)

Let t be a point of T, the fiber over t

is $B(\mathbb{Z}/_2)$ and we have already seen that

$$A^*_\bullet B(\mathbb{Z}/_2) = \overset{0}{\mathbb{Z}} \oplus \overset{-1}{\mathbb{Z}/2} \oplus \overset{-2}{\mathbb{Z}/_2} \oplus \cdots$$

Choose a non-zero $\alpha \in A_i \cdot B(\mathbb{Z}/_2)$ for $i < 0$

We have $j : B(\mathbb{Z}/_2) \to \mathcal{X}$. What is

$$j_* \alpha = ? \quad \text{in} \quad \hat{A}_\bullet \mathcal{X}$$

Since there are no nontrivial v.b. on $\mathcal{X}$ it will be impossible to resolve the non-trivial v.b. over + that gives rise to α.

One solution to this problem can be to declare that $j_*\alpha$ is simply the pair (j, α).

Following this idea we can define $A_\bullet \mathcal{X}$ as all pairs

$$(\mathcal{T} \xrightarrow{f} \mathcal{X}, \beta \in \hat{A}_\bullet \mathcal{T})$$

where f is a projective morphism of

stacks i.e. f factors as

$$\begin{tikzcd} \mathcal{J} \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}(\mathcal{K}) \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

where $\mathbb{P}(\mathcal{K}) = \mathrm{Proj}\,(\mathrm{Sym}\,\mathcal{K})$.

Now for the basic operations with this definition of $A_\circ \mathfrak{X}$ we will get projective pushforwards only!

<u>Remark</u>: The projectivity assumption on f is necessary for the proof of excision.

<u>Def</u>: The Chow groups of an Artin stack $\mathfrak{X}$ are

$$A_i \mathfrak{X} = \varinjlim_{\substack{\text{projective} \\ \mathcal{J} \xrightarrow{f} \mathfrak{X}}} \left(\hat{A}_i \mathcal{J} / \hat{B}_i \mathcal{J} \right)$$

additional equivalences we have to worry about accounting for compatibility of fs from different $\mathcal{J}$'s

Remark: The morphisms in the above limit are inclusions in disjoint unions

$$
\begin{tikzcd}
\mathcal{T} \arrow[r, hook] \arrow[dr] & \mathcal{T} \amalg \mathcal{T}' \arrow[d] \\
& \mathcal{X}
\end{tikzcd}
$$

The groups $A_\bullet \mathcal{X}$ make sense for all Artin stacks and satisfy all the standard properties even over $\mathbb{Z}$. In particular we get $\mathbb{Z}$-valued intersection theory of DM stacks.

Also:

- If X - scheme then $A_\bullet X$ reproduce the usual Chow groups
- If $\mathcal{X}$ - DM stack then $A_\bullet \mathcal{X} \otimes \mathbb{Q}$ are Vistoli's Chow groups
- If $\mathcal{X} = [X/G] \Rightarrow A_\bullet \mathcal{X}$ give the Edidin-Graham definition.

For the IT on Artin stacks we must have also homotopy invariance for vector bundle stacks and this is where one needs the hypothesis that the stabilizers are affine group schemes.

For this one needs the following proposition

<u>Proposition</u>: TFAE

(i) $\mathcal{X}$ has only linear algebraic groups schemes as geometric stabilizers

(ii) $\exists$ a stratification of $\mathcal{X}$ by quotient stacks.