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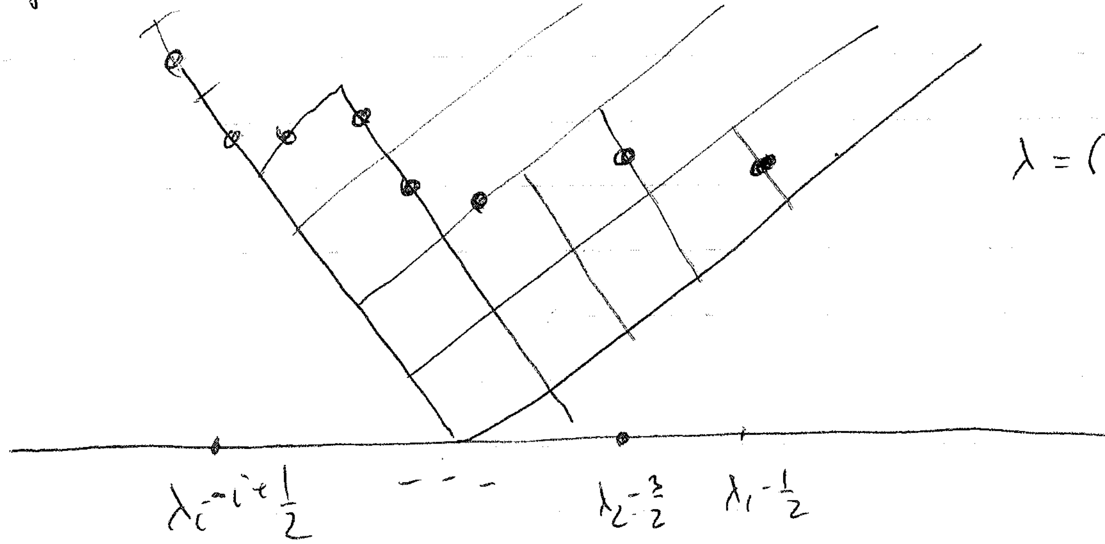
03-12-2002

Gromov-Witten theory of $\mathbb{P}^1$ - part 2

Remark on notation: In the last lecture a non-traditional way of coordinatizing partitions was used:



$\lambda = (5, 3, 1, 1, 0, \ldots)$

For every such partition we looked at

$p_k(\lambda)$ = regularization of the sum $\sum_{i=1}^{\infty} \left(\lambda_i - i + \frac{1}{2}\right)^k$

The regularization of $p_k(\lambda)$ used in the last lecture is

$$p_k(\lambda) = \sum_{i=1}^{\infty} \left[\left(\lambda_i - i + \tfrac{1}{2}\right)^k - \left(-i + \tfrac{1}{2}\right)^k\right] + \left(1 - \frac{1}{2^k}\right) \zeta(-k)$$

$$= k! \, [z^k] \sum_{i=1}^{\infty} e^{z(\lambda_i - i + \frac{1}{2})}$$

<u>Remark</u>: Given a power series $f(z_1, \dots, z_n)$ in variables $z_1, \dots, z_n$ we denote by

$$[z_1^{k_1} \cdots z_n^{k_n}] \, f(z_1, \dots, z_n)$$

the coefficient of f in front of the monomial $z_1^{k_1} \cdots z_n^{k_n}$.

In the notation we just introduced we can rewrite the last formula from Rahul's talk as

$$(*) \quad \left\langle \prod_{i=1}^{n} \tau_{k_i}(p) \right\rangle_d^{\bullet, \mathbb{P}^1} = [z_1^{k_1+1} \cdots z_n^{k_n+1}] \cdot \left(\sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{n} e(\lambda, z_i) \right)$$

where

$$e(\lambda, z) := \sum_{i=1}^{\infty} e^{z(\lambda_i - i + \frac{1}{2})}$$

<u>Remark</u>: If the target for our GW theory is a curve X of genus h we need only replace the exponent 2

in the formula $(*)$ with $2-2h$.

The formula $(*)$ can be understood even better in terms of infinite wedge representations.

Infinite wedge representations:

Let $V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C} \cdot \underline{k}$

Then $\Lambda^{\frac{\infty}{2}} V$ is the collection of all v_λ:

$$\Lambda^{\frac{\infty}{2}} V \ni v_\lambda := \underline{\left(\lambda_1 - \tfrac{1}{2}\right)} \wedge \underline{\lambda_2 - \tfrac{3}{2}} \wedge \underline{\lambda_3 - \tfrac{5}{2}} \wedge \cdots$$

$$v_\emptyset := \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \cdots$$

$\mathfrak{gl}(V) = \mathfrak{gl}(\infty)$ "acts" on $\Lambda^{\frac{\infty}{2}} V$ via

$$\alpha_k \, \underline{i} = \underline{i-k} \qquad \text{for} \quad \alpha_k = \begin{pmatrix} \ddots & & 0 \\ 1 & \ddots & \\ & 1 \;\cdots\; 1 & \ddots \end{pmatrix}_k$$

$$\alpha_{\pm 1} v_\lambda = \sum_{\mu = \lambda \pm \square} v_\mu$$

In fact one checks that

$$[\alpha_k, \alpha_\ell] = k\,\delta_{k+\ell}$$

and so what we get is that a central extension of $gl(\infty)$ acts. Moreover one checks that

$$(\alpha_{-1})^d v_\emptyset = \sum_{|\lambda| = d} (\dim \lambda)\, v_\lambda$$

We can also consider $\mathcal{E}(z) \in gl(V)$ s.t.

$$\mathcal{E}(z) = \begin{pmatrix} \ddots & & & \\ & e^{\frac{3z}{2}} & & 0 \\ & & e^{\frac{z}{2}} & \\ & 0 & & \ddots \end{pmatrix}$$

i.e. $\quad \mathcal{E}(z)\, \underline{i} = e^{zi}\, \underline{i}$

From this formula we get

$$\mathcal{E}(z)\, v_\lambda = e(\lambda, z)\, v_\lambda$$

and so the formula $(*)$ can be rewritten as

$$\left\langle \prod_{i=1}^{n} \tau_{k_i}(p) \right\rangle_d^{\bullet, |P|} = [z_1^{k_1+1} \cdots z_n^{k_n+1}] \cdot$$

$$\cdot \left(\frac{(\alpha_1)^d}{d!} \circ \prod_{i=1}^{n} \mathcal{E}(z_i) \circ \frac{(\alpha_{-1})^d}{d!} v_\emptyset, v_\emptyset \right)$$

where $(\, , \,)$ is the inner product on $\Lambda^{\frac{\infty}{2}} V$ for which all v_λ are orthonormal.

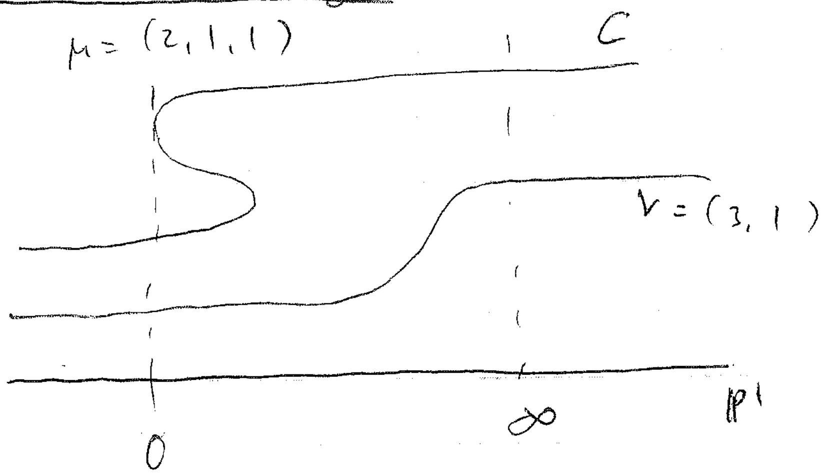
<u>Notation</u>: Given $A \in \mathrm{End}(\Lambda^{\frac{\infty}{2}} V)$ define

$$\langle A \rangle = (A v_\emptyset, v_\emptyset)$$

Thus we get that $(*)$ can be rewritten as

$$\left\langle \prod_{i=1}^{n} T_{k_i}(p) \right\rangle_d^{\bullet, \mathbb{P}^1} = [z_1^{k_1+1} \cdots z_n^{k_n+1}] \cdot \left\langle \frac{(\alpha_1)^d}{d!} \prod_i \mathcal{E}(z_i) \frac{(\alpha_{-1})^d}{d!} \right\rangle$$

<u>Relative theory</u>



$|\mu| = |\nu| = d$

Now

$$\langle \mu \mid \prod T_{k_i}(p) \mid \nu \rangle_d^{\bullet, \mathbb{P}^1} = [z_1^{k_1+1} \cdots z_n^{k_n+1}] \cdot$$

$$\cdot \frac{1}{\mathrm{Aut}\,\mu\ \mathrm{Aut}\,\nu \prod \mu_i \prod \nu_i} \left\langle \prod \alpha_{\mu_i} \prod \mathcal{E}(z_i) \prod \alpha_{-\nu_i} \right\rangle$$

$$\mu = \underbrace{(1, \ldots, 1)}_{d\text{-times}} \qquad \nu = \underbrace{(1, \ldots, 1)}_{d \text{ times}}$$

<u>Example</u>: <u>$n=1$</u>. We need to compute

$$\left\langle \prod \alpha_{\mu_i} \mathcal{E}(z) \prod \alpha_{-\nu_i} \right\rangle^\circ = \quad \leftarrow \text{connected GW invariants}$$

$$= \sum z^{k+1} \langle \mu \mid T_k(p) \mid \nu \rangle^\circ$$

(Remark: $\langle \mu \mid T_k(p) \mid 1^d \rangle^\circ = \dfrac{I^\circ(\mu, \overline{(k+1)})}{k!}$

in the notation from Rahul's talk.)

In order to carry out this computation note first that

$$\alpha_k v_\emptyset = 0 \quad \text{for } k > 0.$$

Now extend the definition of $\mathcal{E}(z)$ to

$\mathcal{E}_\ell(z) \in \mathrm{End}(V)$:

$$\mathcal{E}_\ell(z)\,\underline{i} \stackrel{\mathrm{def}}{=} e^{zi}\,\underline{i-\ell}$$

These operators give $\mathcal{E}_0(z) = \mathcal{E}(z)$
and have commutation relations

$$[\alpha_k, \mathcal{E}_\ell(z)] = \left(1 - e^{-zk}\right)\mathcal{E}_{k+\ell}(z).$$

Then we can commute α_{μ_i} through $\mathcal{E}(z)$ to get

$$\sum_k z^{2g} \langle \mu \,|\, T_k(p) \,|\, \nu \rangle =$$

$$= \frac{1}{\operatorname{Aut}\mu \,\operatorname{Aut}\nu \,\prod \mu_i \prod \nu_i} \cdot \frac{\prod \mathcal{S}(z\mu_i)\,\mathcal{S}(z\nu_i)}{\mathcal{S}(z)}$$

where $\mathcal{S}(z) := \dfrac{\sinh\left(\frac{z}{2}\right)}{\left(\frac{z}{2}\right)}$

If we have an elliptic curve E as a target, then

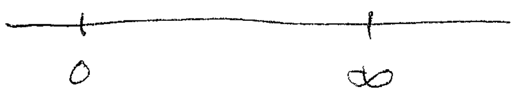
$$\sum_{k_1,\dots,k_n,d} q^d \left\langle \prod_i T_{k_i}(p) \right\rangle_d^{\bullet, E} \prod_i z_i^{k_i+1} = \mathrm{tr}_0\, q^H \prod_i \mathcal{E}(z_i)$$

charge 0 subspace

Here
$$H\,\underline{i} := i\,\underline{i}$$
and so $H\, v_\lambda = |\lambda|\, v_\lambda$.

<u>Remark</u>: The RHS was computed by Bloch–Okounkov in terms of determinants of θ-functions.

<u>Equivariant GW theory of $\mathbb{P}^1$</u>



$[0], [\infty] \in H^\circ_T(\mathbb{P}^1)$

$T = \mathbb{C}^\times$

In fact $H^\circ_T(\mathbb{P}^1)$ is a module over $H^\circ_T(pt) = \mathbb{Z}[t]$ and the identity class
$$\mathbb{1} = \frac{[0]-[\infty]}{t} \in H^\circ_T(\mathbb{P}^1)$$

Now
$$\langle \dots \rangle = \sum_{d,g} u^{2g-2} q^d \langle \dots \rangle_{g,d}$$

and the corresponding generating function is
$$F(z_0, z_1, \dots; w_0, w_1, \dots) = \left\langle \exp\left(\sum_{i=0}^{\infty} \tau_i(0)\, z^i + \tau_i(\infty)\, w^i\right)\right\rangle^{\circ}$$

It turns out that

$$F(z_0, \dots ; w_0, \dots) = \log \left\langle e^{\sum z_k A_{k+1}} \circ e^{\alpha_1} \circ \left(\frac{q}{u^2}\right)^H \circ \right.$$
$$\left. \circ\, e^{\alpha_{-1}} \circ e^{\sum_k w_k A^*_{k+1}} \right\rangle$$

for some explicit

$$A_k = \begin{pmatrix} \ddots & \tau_k \\ & \ddots \end{pmatrix}, \qquad A_k^* := A_k|_{t \to -t}$$

In fact, already the existence of a formula like that (regardless of the explicit form of the A_k's) implies

<u>Corollary</u> (Toda equation)

$$\frac{\partial^2}{\partial z_0 \partial w_0} F = e^{\Delta F}$$

where $\Delta F = F|_{s \to s+u} + F|_{s \to s-u} - 2F$

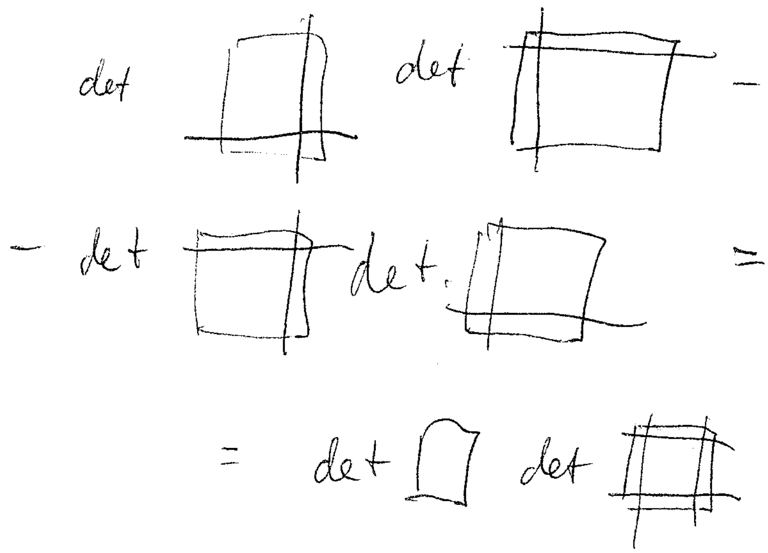
and

$$\frac{\partial}{\partial s} = \text{conj to } \mathbb{1} = \frac{[0] - [\infty]}{t}$$

i.e. $\frac{\partial}{\partial s} F = \langle \tau_0(\mathbb{1}) \exp(\dots) \rangle$

<u>Remark</u>: $\frac{\partial}{\partial s} = \frac{1}{t}\left(\frac{\partial}{\partial z_0} - \frac{\partial}{\partial w_0}\right)$.

<u>Remark</u>: • The above Toda equation corresponds to the lowest order Plücker identity in the semi-infinite matrix F:



(One has to rewrite this identity for 1st + 2nd rows of a matrix rather than 1st + last, so that it will make sense for infinite matrices)

• All the other Plücker identities give rise to higher equations in the Toda hierarchy.

To write down A_k explicitly we look at a generating function for the A_k's:

$$A(z) = \sum_{k \in \mathbb{Z}} z^k A_k = \frac{1}{4} \mathcal{J}(uz)^{tz}$$

$$= \sum_{k \in \mathbb{Z}} \frac{(1-e^{-uz})^k}{(1+tz)\cdots(1+kz)} \, \xi_k(uz).$$

<u>Note</u>: In terms of the A_k we can write the n-point functions as

$$\left\langle \prod A(z_i)\, e^{\alpha_1} \left(\frac{q}{u^2}\right)^{H} e^{-\alpha_1} \prod A^{\star}(w_i) \right\rangle =$$

$=$ multiple hypergeometric sum