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Toric stacks of pavings

This talk is the first of three giving a general procedure for compactifying moduli spaces in projective geometry.

Motivation:

Consider a graded vector space

$$E = E_0 \oplus \cdots \oplus E_n \qquad \dim E_i < +\infty$$

Consider:

- $Gr^{r,E} = \{F \hookrightarrow E \mid \dim F = r\}$
- $\forall\, I \subset \{0, 1, \ldots, n\}$ put
  $$E_I := \bigoplus_{\alpha \in I} E_\alpha$$
- given $F \in Gr^{r,E}$, $I \subset \{0, 1, \ldots, n\}$

define

$$d_I(F) = \dim(F \cap E_I)$$

These numbers satisfy

$$d_\emptyset = 0, \quad d_{\{0,\ldots,n\}} = r$$

$$d_I + d_J \leq d_{I\cap J} + d_{I \cup J} \qquad \forall\, I, J$$

A collection of numbers $\{d_I\}_I$ satisfying the above properties is called a <u>matroid of rank r over $\{0,1,\dots,n\}$</u>.

Given a matroid $\{d_I\}_I$ consider

$$Gr^{r,E}_{(d_I)} = \{F \in Gr^{r,E} \mid \dim(F \cap E_I) = d_I\}$$

Then $Gr^{r,E}_{(d_I)} \subset Gr^{r,E}$ is a closed subscheme

Moreover $Aut(E_0) \times \dots \times Aut(E_n)$ acts on $Gr^{r,E}_{(d_I)}$ and if we look at the action of $\mathbb{G}_m^{\times n+1}$ inside $Aut(E_0) \times \dots \times Aut(E_n)$
$\Rightarrow$ can form

$$\overline{Gr}^{r,E}_{(d_I)} = Gr^{r,E}_{(d_I)} / \mathbb{G}_m^{n+1}$$

$\uparrow$ the <u>thin Schubert cell</u>

<u>Examples</u>:

(1) $E_\alpha = \mathbb{A}^r \quad \forall \alpha$

$d_I = 0, \quad \forall I \subsetneq \{0,1,\dots,n\}$

$$\overline{Gr}^{r,E}_{(d_I)} = PGL_r^{n+1} / PGL_r$$

(2) $\quad \mathrm{rk}\ E_\alpha = 1 \quad \forall\, \alpha$

$\overline{Gr}^{r,E}_{(d_I)}$ = configuration space of $n+1$ points in $\mathbb{P}^{r-1}$

<u>Thales</u>: Any integral scheme of finite type over $\mathbb{Z}$ contains an open $\cong \overline{Gr}^{3,E}_{(d_I)}$.

<u>Main problem</u>: Compactify $\overline{Gr}^{r,E}_{(d_I)}$.

Our approach to this problem will involve the following steps:

- Construct a bijection

$$\begin{pmatrix} \text{arbitrary matroids} \\ (d_I) \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Convex polytope} \\ \mathcal{S} \end{pmatrix}$$

- Construct a compactification

$$\overline{\Omega}^{r,E}_{\mathcal{S}} \text{ of } \overline{Gr}^{r,E}_{(d_I)} = \overline{Gr}^{r,E}_{\mathcal{S}}$$

endowed with a morphism

$$\overline{\Omega}^{ne}_{S} \longrightarrow \mathcal{A}^{S} / \mathcal{A}^{S}_{\emptyset}$$

where
$\mathcal{A}^{S} / \mathcal{A}^{S}_{\emptyset}$ is a toric stack
i.e. a quotient of
a toric variety by
its torus

Moreover we will show that the points of $\mathcal{A}_S / \mathcal{A}^S_{\emptyset}$ are in a bijection with all the pavings of the polytope S.

<u>Remark</u>: • Every morphism
Scheme $\longrightarrow$ toric stack
can be interpretted as a log structure on the scheme
• Conversely (Martin Olsson)
every log structure on a scheme can be locally interpretted as a morphism to a toric stack.

Next week we will give two different modular interpretations of $\overline{\Omega}^{ne}_{S}$. For these we will need another toric stack

$$\hat{A}^{S} / \hat{A}^{S}_{\emptyset} \longrightarrow A^{S} / A^{S}_{\emptyset}$$

So that the points of $\hat{A}^{S} / \hat{A}^{S}_{\emptyset}$ correspond to pairs

(paving of S, a face of the paving).

Today we will explain what $A^{S} / A^{S}_{\emptyset}$ and $\hat{A}^{S}_{S} / \hat{A}^{S}_{\emptyset}$ are.

Integral polytopes

$$S^{r,n} = \{ \underline{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + \dots + i_n = r \}$$

$$S^{r,n}_{\mathbb{R}} = \{ \underline{x} = (x_0, \dots, x_n) \in (\mathbb{R}^{+})^{n+1} \mid x_0 + \dots + x_n = r \}$$

Given a matroid $\{d_I\}_I$ consider

$$S := \{ \underline{i} \in S^{r,n} \mid \sum_{\alpha \in I} i_\alpha \geq d_I, \ \forall I \}$$

$$S_{\mathbb{R}} := \{ \underline{x} \in S^{r,n}_{\mathbb{R}} \mid \sum_{\alpha \in I} x_\alpha \geq d_I, \ \forall I \}$$

Then $(S_{\mathbb{R}}, S)$ is the integral polytope corresponding to the matroid $\{d_I\}_I$

<u>Proposition</u>: The assignment
$$(d_I)_I \rightsquigarrow (\mathcal{S}_{\mathbb{R}}, \mathcal{S})$$
Satisfies the following properties

(i) $\forall\ I \subseteq \{0, 1, \dots, n\}$
the numbers d_I can be reconstructed from the pair $(\mathcal{S}, \mathcal{S}_{\mathbb{R}})$ by

$$d_I := \min \{ \sum_{\alpha \in I} i_\alpha \mid \underline{i} \in \mathcal{S} \}$$

i.e. $\mathcal{S}$ reconstructs $\{d_I\}_I$

(ii) $\mathcal{S}_{\mathbb{R}}$ is generated by $\mathcal{S}$ as a convex polytope

(iii) The faces of $\mathcal{S}_{\mathbb{R}}$ are also integral polytopes (i.e. correspond to other matroids)

(iv) Any polytope which admits a paving by integral polytopes is again an integral polytope

(v) If $\dim \mathcal{S}_{\mathbb{R}} = n-p$, then there exists a unique decomposition

$$\{0, 1, \dots, n\} = J_0 \amalg \dots \amalg J_p, \quad |J_i| = n_i + 1$$

$$r = r_0 + r_1 + \cdots + r_p$$

$$S_{\mathbb{R}} = S^0_{\mathbb{R}} \times \cdots \times S^p_{\mathbb{R}}$$
$$S = S^0 \times \cdots \times S^p$$

S^i = the integral polytope of maximal dimension in $S^{r_i n_i}$

$$= \{ (i_\alpha)_{\alpha \in J_i} \in \mathbb{N}^{J_i} \mid \sum_{\alpha \in J_i} i_\alpha = r_i \}$$

(vi) Any integral polytope of maximal dimension contains a family of points which generates the lattice of maximal points

Proof:

$$\emptyset \subsetneq I_0 \subsetneq \{0, 1, \ldots, n\}$$
$$J_0 = \{0, 1, \ldots, n\} - I_0$$

$$S_{I_0} = \{ (i_0, \ldots, i_n) \in S \mid \sum_{\alpha \in I_0} i_\alpha = d_{I_0} \}$$

$\forall \; I \subset \{0, 1, \ldots, n\}$

$$d_I \leq d_{I \cap I_0} + (d_{I \cup I_0} - d_{I_0})$$

$$S_{I_0} = S'_{I_0} \times S''_{J_0}$$

$$S'_{I_0} \subseteq \{ (i_\alpha)_{\alpha \in I_0} \mid \sum i_\alpha = d_{I_0} \}$$

is defined by the matroid

$$(I \subset I_0) \longmapsto d'_I = d_I$$

$$S''_{I_0} \subseteq \{ (i_\alpha)_{\alpha \in J_0} \mid \sum i_\alpha = r - d_{I_0} \}$$

is defined by the matroid

$$(I \subset J_0) \mapsto d''_I = d_{I \cup I_0} - d_{I_0}.$$

Now we can prove (i) – (vi) by induction on the dimension

□.

Two toric varieties A^S, $\tilde{A}^S$:

Let $S \subseteq S^{r,n}$ be an integral polytope

Let $\underline{S}$ = a paving of S by integral polytopes

Given a paving $\underline{S}$ we may consider

$$C^S_{\underline{S}} \subset \mathbb{R}^S$$

where

$\mathcal{C}^{\mathcal{S}}_{\underline{\mathcal{S}}}$ = cone of (convex?) functions

$v : \mathcal{S} \longrightarrow \mathbb{R}$

such that for all $\mathcal{S}'$ - cell of $\underline{\mathcal{S}}$

there exists an affine function

$$l : \mathcal{S} \to \mathbb{R}$$

s.t.

- $l \leq v$
- $\mathcal{S}' = \{ \underline{i} \in \mathcal{S} \mid l(\underline{i}) = v(\underline{i}) \}$

Def: We will say that $\underline{\mathcal{S}}$ is an integral convex paving of $\mathcal{S}$
$\Longleftrightarrow$ $\mathcal{C}^{\mathcal{S}}_{\underline{\mathcal{S}}} \neq \emptyset$.

If $\underline{\mathcal{S}}$ is a paving of $\mathcal{S}$ and $\mathcal{S}'$ is a face of $\underline{\mathcal{S}}$ then we can associate to $(\underline{\mathcal{S}}, \mathcal{S}')$ another cone:

$\tilde{\mathcal{C}}^{\mathcal{S}}_{\underline{\mathcal{S}}, \mathcal{S}'}$:= cone of all $v : \mathcal{S} \longrightarrow \mathbb{R}$ s.t.
$v \in \mathcal{C}^{\mathcal{S}}_{\underline{\mathcal{S}}}$ and s.t.

$$\mathcal{S}' = \{ \underline{i} \in \mathcal{S} \mid v(\underline{i}) = \min(v) \}$$

Example: If $\emptyset$ is the trivial paving of S, then

$$\mathcal{C}^S_\emptyset = \text{subspace of all affine functions } f : S \longrightarrow \mathbb{R}$$

Hence

$$\mathcal{C}^S_{\underline{S}} + \mathcal{C}^S_\emptyset = \mathcal{C}^S_{\underline{S}}$$

for all other paving $\underline{S}$ of S.
Similarly

$$\mathcal{C}^S_{\underline{S}, S'} + \mathbb{R} = \tilde{\mathcal{C}}^S_{\underline{S}, S'}$$

Proposition: (i) $\left(\mathcal{C}^S_{\underline{S}} / \mathcal{C}^S_\emptyset \right) \subset \mathbb{R}^S / \mathcal{C}^S_\emptyset$

(ii) $\left(\tilde{\mathcal{C}}^S_{\underline{S}, S'} / \mathbb{R} \right) \subset \mathbb{R}^S / \mathbb{R}$

are fans.

Consider the associated toric varieties

$\mathcal{A}^{S}$ - with torus $\mathcal{A}^{S}_{\emptyset} = \mathbb{G}_m^{S} / (\mathbb{G}_m^{S})_{\emptyset}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

affine functions $S \to \mathbb{G}_m$

$\tilde{\mathcal{A}}^{S}$ - with torus $\tilde{\mathcal{A}}^{S}_{\emptyset} = \mathbb{G}_m^{S} / \mathbb{G}_m$

Moreover

$$\begin{pmatrix} \text{integral convex} \\ \text{pavings } \underline{S} \text{ of } S \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Orbits } \mathcal{A}^{S}_{\underline{S}} \text{ of} \\ \mathcal{A}^{S} \end{pmatrix}$$

$$\begin{pmatrix} \text{Pairs } (\underline{S}, S') \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{orbits } \tilde{\mathcal{A}}^{S}_{\underline{S}, S'} \text{ of} \\ \tilde{\mathcal{A}}^{S} \end{pmatrix}$$

<u>Proposition</u>:

(i) $\mathcal{A}^{S}$ and $\tilde{\mathcal{A}}^{S}$ are quasi projective

(ii) there is a natural forgetful equivariant morphism

$$\tilde{\mathcal{A}}^{S} \to \mathcal{A}^{S}$$

(for $\tilde{\mathcal{A}}^{S}_{\emptyset} = \mathbb{G}_m^{S} / \mathbb{G}_m \twoheadrightarrow \mathcal{A}^{S}_{\emptyset} = \mathbb{G}_m^{S} / (\mathbb{G}_m^{S})_{\emptyset}$)

This morphism is projective and flat of relative dimension $= \dim \underline{S}$.

The fibers are geometrically reduced

$\mathbb{A}^{\underline{S}}_{\underline{S}} \ni \alpha_{\underline{S}}$

$Y_{\underline{S}} = (\tilde{\mathbb{A}}^{\underline{S}} \times_{\mathbb{A}^{\underline{S}}} \alpha_{\underline{S}})$

$\circlearrowleft$

$(\mathbb{G}_m^{\underline{S}})_\phi \twoheadleftarrow \mathbb{G}_m^{n+1}$

- $Y_{\underline{S}}$ – projective scheme of dimension $= \dim \underline{S}$
- finite number of orbits $Y_{\underline{S}'} \Leftrightarrow$ faces $\underline{S}'$ of $\underline{S}$.
- $Y_{\underline{S}'}$ = normal projective torus variety of dim $= \dim \underline{S}'$.
- $Y_{\underline{S}''} \subset \overline{Y_{\underline{S}'}} \Leftrightarrow \underline{S}'' =$ face of $\underline{S}'$.