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Logarithmic geometry and algebraic stacks

The connection between log geometry and algebraic stacks was first studied by Illusie (following a suggestion of Lafforgue). This approach uses toric stacks i.e. the quotients of a toric varieties by their tori. The approach we will be interested in will be different.

Basic idea: Many interesting morphisms fit naturally into a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\text{not smooth}"'] \arrow[dr, "\text{smooth}"] & \\ S & \mathcal{S} \arrow[l, "\text{relative dim } 0"] \end{tikzcd}$$

$\mathcal{S}$ ← alg. stack

Outline: (1) Review of log geometry

(2) Define $\mathcal{L}og_S$ and explain the relation to toric stacks

(3) Log smoothness

(4) $\mathcal{M}_g$.

1. Log geometry

Def: A log structure on a scheme X is a pair

$$(\mathcal{M}, \alpha)$$

consisting of

- $\mathcal{M}$ – a sheaf of commutative monoids on X_{et}
- homomorphism of monoids

$$\alpha : \mathcal{M} \longrightarrow (\mathcal{O}_X, \cdot)$$

such that

$$\alpha^{-1}(\mathcal{O}_X^{\times}) \xrightarrow{\sim} \mathcal{O}_X^{\times}$$

Example. Let X be an integral scheme
$U \subseteq X$ – open

Consider $\mathcal{M}_X := \{ f \in \mathcal{O}_X \mid f_{|U} \text{ - unit} \}$
then

$$\mathcal{M}_X \hookrightarrow \mathcal{O}_X$$

is a log structure on X.

The log scheme $(X, M_X \hookrightarrow \mathcal{O}_X)$ should be thought of as the open scheme U + the way U sits inside X.

<u>Remark</u>: If we have any monoid M + a map $M \to \mathcal{O}_X$ (not necessarily satisfying the condition on units) $\Rightarrow$ we can promote M to a log structure by taking

$$N = \mathcal{O}_X^\times \oplus M / \sim \longrightarrow \mathcal{O}_X.$$

This construction gives many other examples, e.g.

<u>Example</u>: Let P be any finitely generated monoid and let

$$X = \mathrm{Spec}(\mathbb{Z}[P])$$

Then $P \to \mathcal{O}_X$ generates (in the sense of the remark above) a log structure $M_X \to \mathcal{O}_X$.

<u>Def:</u> The category of log structures on X is the category with

$$\mathrm{Ob} = (X, \mathcal{M}, \alpha) \text{ - log scheme}$$

$$\mathrm{Mor} = \begin{tikzcd} \mathcal{M} \arrow[rr] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{M}' \arrow[dl, "\alpha'"] \\ & \mathcal{O}^{\times} & \end{tikzcd}$$

morphism of monoids.

<u>Example:</u> If $f : X \to Y$ is a morphism of schemes and if $\mathcal{M}_Y$ is a log structure on Y $\Rightarrow$ we can define a log structure

$$f^{*}\mathcal{M}_Y \text{ on } X$$

as the log structure corresponding to the monoid $f^{-1}\mathcal{M}_Y$, i.e.

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r] \arrow[dr] & f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \\ & f^{*}\mathcal{M}_Y \arrow[ur, dashed] & \end{tikzcd}$$

<u>Def:</u> A morphism of log schemes is a pair

$$(X, M_X) \xrightarrow{(f, f^b)} (Y, M_Y)$$

s.t.

• $f: X \to Y$ is a morphism of schemes

• $f^b: f^* M_Y \to M_X$ - morphism of log structures on X.

<u>Def:</u> If X is a log scheme, then X is <u>fine</u> if etale locally on X there exists a morphism

$$X \xrightarrow{(f, f^b)} (\operatorname{Spec} \mathbb{Z}[P], \text{natural log structure})$$

for some P - f.generated, so that f^b is an isomorphism.

<u>Remark:</u> • $(\operatorname{Spec} \mathbb{Z}[P]$, nat log str.$)$ as above is called a chart for X.

• From now on all log structures we will look at will be fine.

The stack $\mathcal{L}og_{\mathcal{S}}$

Let $\mathcal{S}$ be a fixed fine log scheme

Let $\underline{\mathcal{S}}$ be the underlying scheme

We consider a fibered category

$$\mathcal{L}og_{\mathcal{S}} \longrightarrow (\underline{\mathcal{S}}\text{-schemes})$$

$$X/\mathcal{S} \longrightarrow \underline{X}/\underline{\mathcal{S}}$$

Where

$\mathrm{Ob}(\mathcal{L}og_{\mathcal{S}}) = X \to \mathcal{S}$ - morphism of fine log scheme

$\mathrm{Mor}(\mathcal{L}og_{\mathcal{S}})$:

$$\begin{tikzcd} X' \arrow[rr, "(f, f^b)"] \arrow[dr] & & X \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}$$

- morphisms of log schemes s.t. f^b is an isomorphism.

<u>Example:</u> If we take $\underline{S} = (\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}^\times)$ – the initial object in the category of log schemes, then if we put

$$Log := Log_{\underline{S}}$$

$$\Rightarrow \quad Log(T) = \text{groupoid of log structures on } T.$$

<u>Theorem (Olsson)</u> $Log_{\underline{S}}$ is an algebraic stack which is locally of finite presentation on $\underline{S}$.

<u>Remark:</u> $\Delta : Log_{\underline{S}} \to Log_{\underline{S}} \times Log_{\underline{S}}$ is not separated but is locally separated.

A presentation of Log

$$P \rightsquigarrow \left(\mathrm{Spec}\,\mathbb{Z}[P],\ \begin{matrix}\text{canonical} \\ \text{log structure}\end{matrix}\right)$$

$$\| \! \wr$$

$$D(P)$$

<u>Theorem</u>: $\coprod_{P} \mathrm{Spec}(\mathbb{Z}[P]) \to \mathcal{L}og$

is a flat cover of $\mathcal{L}og$ which is smooth if the group P^{g} associated to the monoid P is torsion free.

In fact we can make this more precise.

Consider $\underline{D}(P) := \mathrm{Spec}(\mathbb{Z}[P])$

Then

$D(P^{g})$ acts on $\underline{D}(P)$

since

$$\underline{D}(P)(R) = \mathrm{Hom}_{\text{Monoids}}(P, R)$$

$$\cup\!|$$

$$D(P^{g})(R) = \mathrm{Hom}_{\text{Groups}}(P^{g}, R^{\times})$$

Consider the toric stack

$$\mathcal{S}_{P} := [\underline{D}(P) / D(P^{g})]$$

then we have

<u>Theorem</u>: $\mathcal{S}_{\underline{P}} \to \mathcal{L}og$ is etale.

<u>Remark</u>: $\mathcal{S}_{\underline{P}} \to \mathcal{L}og$ is not an isomorphism

<u>Example</u>: $\underline{P} = \mathbb{N}^r$
$\underline{D(P)} = \mathbb{A}^r \circlearrowleft \mathbb{G}_m^r = D(\underline{P})$

In fact we can identify $\mathcal{S}_{\underline{P}}$ with the classifying stack of pairs (M, β) where

M - log structure on $\operatorname{Spec} \mathbb{Z}$

$\beta : \underline{P} \to M/\mathcal{O}^\times$ - morphism of monoids which locally lift to a chart

<u>Example</u>: $[\mathbb{A}^r / \mathbb{G}_m^r] = \{ (M, \beta) \mid \mathbb{N}^r \xrightarrow{\beta} M/\mathcal{O}^\times$

<u>Log Smoothness</u>

Let $\mathcal{P}$ be a property of a representable morphism of stacks

Def: If $f : X \to S$ is a morphism of log schemes $\Rightarrow$ we say that f has property $\mathcal{P}$ if the natural map

$$\underline{X} \to \mathcal{L}og_S$$

has the property $\mathcal{P}$.

Example: If R is a DVR with a uniformizing parameter π $\Rightarrow$ if $K = \mathrm{Frac}(R)$ we may look at a semistable curve

$$\begin{array}{ccc} C_K & \subset & C \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \subset & \mathrm{Spec}(R) \end{array}$$

$\Rightarrow$ the natural local map

$$\begin{array}{c} \mathrm{Spec}\left(R[x,y]/xy-\pi \right) \\ \downarrow \\ \mathrm{Spec}\, R \end{array}$$

gives rise to a natural morphism of log schemes

$$\begin{tikzcd} (C, \mathcal{M}_C) \arrow[d] \\ (\mathrm{Spec}(R), \mathcal{M}_R) \end{tikzcd}$$

$$x \longleftarrow (1,0)$$
$$y \longleftarrow (0,1)$$

$$\begin{tikzcd} \mathrm{Spec}\left(R[x,y]/xy-\pi\right) \arrow[r, "\text{Chart}"'] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & D(\mathbb{N}^2) \arrow[d] & \mathbb{N}^2 \\ \mathrm{Spec}(R) \arrow[r, "\text{Chart}"'] & D(\mathbb{N}) & \mathbb{N} \arrow[u, "\Delta"'] \end{tikzcd}$$

$$\pi \longleftarrow 1$$

In fact this gives

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & D(\mathbb{N}^2) \arrow[d] \\ \mathcal{L}og_{\mathrm{Spec} R, \mathcal{M}_R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & \mathcal{L}og_{D(\mathbb{N})} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & D(\mathbb{N}) \end{tikzcd}$$

__Exercise__: • Show that

$$\begin{tikzcd} C \arrow[r] \arrow[dr, "\text{smooth}"'] & \operatorname{Spec} R \\ & \mathrm{Log}_{\operatorname{Spec} R,\, M_R} \arrow[u] \end{tikzcd}$$

• Show that the relative differentials for

$$C \to \mathrm{Log}_{\operatorname{Spec} R,\, M_R}$$

are exactly the differentials on C with logarithmic poles along the closed fiber.

$\overline{\mathcal{M}}_g$

The stack $(\overline{\mathcal{M}}_g, M_\infty)$ is now a log stack (with the log structure defined above)

In particular $(\overline{\mathcal{M}}_g, M_\infty)$ will be a fibered category over the category of log schemes

We will define now a fibered category $\mathcal{M}_g^{\log}$ over the category of log schemes independently.

Objects /T : morphisms of log schemes
$f: C \to T$
s.t.
(1) f log smooth proper vertical, integral
(2) $\forall\ \bar{t} \to T \Rightarrow$
$C_{\bar{t}} \to \bar{t}$
satisfies
(i) $\underline{C}_{\bar{t}}$ - reduced curve of genus g
(ii) $\underline{C}_{\bar{t}} \to \mathcal{L}og_{\bar{t}}$
is s.t.

If $\left(\Omega^1_{C_{\bar{t}} / \mathcal{L}og_{\bar{t}}} \right)^{\vee} =: \Theta_{C_{\bar{t}}}$

then
$H^0(C_{\bar{t}}, \Theta_{C_{\bar{t}}}) = 0.$

<u>Theorem</u>: $\mathcal{M}_g^{\log}$ is equivalent to the log stack defined by $(\overline{\mathcal{M}}_g, \mathcal{M}_{\infty})$.

The proof involves two steps

Step 1: $\exists$ fibered category $(F, \mathcal{M}_F)$ such that

$$\mathcal{M}_g^{\log} \simeq \text{associated stack}$$

Step 2: the fibered product

$$\begin{tikzcd} F \arrow[d, "\mathcal{M}_F"'] & F \times_{\mathcal{L}og} \mathbb{Z}[P] \arrow[l] \arrow[d] \\ \mathcal{L}og & \operatorname{Spec} \mathbb{Z}[P] \arrow[l] \end{tikzcd} \qquad P = \mathbb{N}^r$$

is algebraic

Now note that the defo theory of $F \times_{\mathcal{L}og} \mathbb{Z}[P]$ is easy to understand

Indeed given R - $\mathbb{Z}[P]$ algebra and solid arrows

$$\begin{tikzcd} C \arrow[r, hook, dashed] \arrow[d] & C' \arrow[d, dashed] \\ (\operatorname{Spec} R, \mathcal{M}_R) \arrow[r, hook] & (\operatorname{Spec} R', \mathcal{M}_{R'}) \end{tikzcd}$$

we need to find C' and the dotted arrows

This can be rewritten for

$$\begin{tikzcd} \underline{C} \arrow[r, hook, dashed] \arrow[d] \arrow[dr, phantom, "\boxtimes"] & \underline{C}' \arrow[d] \\ \mathcal{L}og(R, M_R) \arrow[r, hook] & \mathcal{L}og(R', M_{R'}) \end{tikzcd}$$

The deformation is given by

$$H^i(\underline{C}, \Theta_{\underline{C}/\mathcal{L}og(\operatorname{Spec} R, M_R)})$$