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How many GW invariants of rational curves are there really?

- When do GW invariants bend and break along with rational curves?
- What are the GW invariants of G/B?

Recall: We have $\overline{M}_{g,n}(X,\beta)$ - moduli of stable maps and

$$ev_{g,n} : \overline{M}_{g,n}(X,\beta) \longrightarrow X^{\times n}$$

+ GW invariants $ev_{g,n}(\mathbb{1}_\beta) \in H^\bullet(X^{\times n})$.

We will mainly look at $g=0$.

Thm (Kontsevich-Manin) If $H^\bullet(X)$ is generated by divisor classes, then

$$ev_{0,2*}(\mathbb{1}_\beta)$$

determine all the rest of GW invariants

Suppose $H^\bullet(X) \cong \mathbb{Q}[H]/H^{n+1}$

$K_X \sim -f \cdot H$

Quantum cohomology : $QH^\bullet(X)$

$$H * H^n = a_{n,1}\, q\, H^{n+1-f} + a_{n,2}\, q^2 H^{n+1-2f}$$

$$H * H^{n-1} = H^n + a_{n-1,1}\, q H^{n-f} + \dots$$

All the coefficients $a_{n,i}$ above are values of $ev_{0,2*}$ on different $\mathbb{1}_\beta$'s.

<u>Goal</u>: Compute $a_{n,i}$ $\forall i$ in terms of $ev_{0,2*}(\mathbb{1}_{line})$ only.

Alternative point of view

Choose $X \subset \mathbb{P}^n$ s.t. $\beta \to d(\text{line})$

Look at $\overline{M}_{0,0}(X \times \mathbb{P}^1, (\beta, 1))$

$$\overline{M}_{0,0}(X \times \mathbb{P}^1, (\beta,1)) = \overline{Map_\beta(\mathbb{P}^1, X)} \quad \text{graph space}$$

In fact

$$\overline{Map_\beta(\mathbb{P}^1, X)} \subset \overline{Map_d(\mathbb{P}^1, \mathbb{P}^n)}$$

$$\downarrow$$

$$\mathbb{P}\left(\begin{matrix}\text{homogeneous}\\ \text{polynomials}\end{matrix}\right)$$

We can now look at the map

$$\overline{M}_{0,0}(X \times \mathbb{P}^1, (\beta, 1)) \xrightarrow{ev} \mathbb{P}(\text{polys})$$

which is clearly $\mathbb{C}^\times$ equivariant and so as a substitute we can look at

$$ev_*^{\mathbb{C}^\times}(\mathbb{1}_\beta^{\mathbb{C}^\times}) \in H^\bullet_{\mathbb{C}^\times}(\mathbb{P}(\text{polys}))$$

But all classes in $\mathbb{P}(\text{polys})$ can be localized along the $\mathbb{C}^\times$ fixed points. Also the fixed loci are easy to describe - they are spans of monomials, i.e. each component of the fixed locus is

$$\mathbb{P}_k = \mathbb{P}(\text{Span}(\alpha_0 x^k y^{d-k}, \dots, \alpha_n x^k y^{d-k}))$$
$$\cong \mathbb{P}^{n-1}$$

$$\Rightarrow ev_*^{\mathbb{C}^\times}(\mathbb{1}_\beta^{\mathbb{C}^\times}) = \sum_k i_{k*}\left(\frac{i_k^*(\mathbb{1}_\beta^{\mathbb{C}^\times})}{e_{\mathbb{C}^\times *}(\mathbb{P}_k)}\right)$$

Thm (Lee-Pandharipande, Bertram-Kley)

If $H^*(X)$ is generated by divisors then

$$ev_{1*}\left(\frac{\mathbb{1}_\beta}{\hbar(\hbar-\psi)}\right)$$

determine all the GW invariants.
Here $\hbar$ is an equivariant parameter.

Remark: $ev_{1*}\left(\frac{\mathbb{1}_\beta}{\hbar(\hbar-\psi)}\right)$ has the same # of parameters for each β as do $ev_{2*}(\mathbb{1}_\beta)$.

Hope: If X is Fano with the property that

$$-K_X \cdot \beta \geq 2$$

for all rational curves and if $H^*(X) = \mathbb{Q}[H]/H^{n+1}$, then

$$j_\beta(H,\hbar) = j_\alpha(H,\hbar)\, j_{\beta-\alpha}(H+(\alpha\cdot H)\hbar,\hbar)$$

<u>Examples:</u> • $X = \mathbb{P}^n$, $ev_{2*}(\mathbb{1}_e) = 1$

$$ev_{1*}\left(\frac{\mathbb{1}_e}{\hbar(\hbar-\psi)}\right) = \frac{1}{(H+\hbar)^{n+1}}$$

$$ev_{2*}\left(\frac{\mathbb{1}_{2e}}{\hbar(\hbar-\psi)}\right) = \frac{1}{(H+\hbar)^{n+1}(H+2\hbar)^{n}}$$

• If $X \subset \mathbb{P}^n$ - Fano hypersurface of index ≥ 2

$$ev_{1*}\left(\frac{\mathbb{1}}{\hbar(\hbar-\psi)}\right) = \frac{(H+\hbar)\cdots(H+\hbar)}{(H+\hbar)^{n+1}}$$

$$ev_{1*}\frac{\mathbb{1}_2}{\hbar(\hbar-\psi)} = j_1(H,\hbar)\, j_1(H+\hbar, 2)$$