Nonemptiness of symmetric degeneracy loci (MSRI talk 3/15/02)

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Nonemptiness of symmetric degeneracy loci

I. Introduction

Q: Suppose X = projective variety ($/\mathbb{C}$ in this talk), Y = closed subvariety.

When is the following true: If $Z \subset X$ is closed and $\dim Z \geqslant \operatorname{codim}_X Y$, then $Z \cap Y$ is nonempty?

$X = \mathbb{P}^n$ ✓

$X = \mathrm{Flags}(\mathbb{C}^3) = \{ E_1 \subset E_2 \subset E_3 = \mathbb{C}^3 \}$?

False: it's well-known that $H^*X \cong \mathbb{Z}[x_1, x_2, x_3] / \mathbb{Z}[\ ]_+^W$, $x_i = -c_1(E_i / E_{i-1})$; there is a 2-dimensional Schubert variety Y and a 1-dim'l Schubert variety Z with

$$[Y] = x_1^2 \cap [X] \qquad [Z] = x_1 \cap [X]$$

We can move Y and Z to general position so $x_1^3 \cap [X]$ = # of pts in $Y \cap Z$; but (exercise: $x_1^3 = 0$ in H^*X so this number is 0.

<u>Example</u>. Consider the variety $\mathbb{P}(S_N(\mathbb{C}))$, $S_N(\mathbb{C})$ = symmetric $N \times N$ matrices.

Fix $r \leq N$. Let $X = X_r$ = subvariety of $\mathbb{P}(S_N(\mathbb{C}))$ corresponding to matrices of rank $\leq r$.

Let $Y = X_{r-1} \subseteq X$. Then $\operatorname{codim}_X Y = N - r + 1$. If $Z \subseteq X$ has dimension $\geqslant N-r+1$ must $Z \cap Y$ be nonempty?

<u>A</u>: Yes. (Ilic–Landsberg: r even, Z smooth and simply connected)

Observe that the trivial rank N bundle $V = X \times \mathbb{C}^N \longrightarrow X$ has a tautological quadratic form with values in the line bundle $\mathcal{O}_X(1)$. The key point here:

$S^2(V^*) \otimes \mathcal{O}_X(1)$ is an ample vector bundle

__Theorem__ Let $V \to X$ be a vector bundle of rank N on a complex projective scheme of dimension d. Suppose that V is equipped with a quadratic form of constant rank r ($r \leq N$) with values in a line bundle L, such that $S^2(V^*) \otimes L$ is ample. Then $d \leq N-r$.

[Equivalently: if $V \to X$ is a rank N bundle equipped with an L-valued quadratic form of rank $\leq r$ at every point, and $S^2(V^*) \otimes L$ is ample, and $\dim X > N-r$, then $X_{r-1} = \{x \mid \text{rank quad}_x \leq r-1\}$ is nonempty]

Applied to $X = X_r \subseteq \mathbb{P}(S_N(\mathbb{C}))$, $Y = X_{r-1}$, this gives the result stated in the above example.

II. Quadric bundles

1. Cohomology of the quadric

Assume $V = \mathbb{C}^N$ ($N = 2n$ or $2n+1$), equipped with a nondegenerate quadratic form. Let $Q \subset P(V)$ denote the quadric of isotropic lines. If E and F are maximal isotropic subspaces of V (necessarily of dim n) then $P(E)$ and $P(F)$ are subvarieties of Q called rulings. $[P(E)] = [P(F)] \iff E$ and F are in the same orbit of $SO(V)$ on the Grassmannian of isotropic n planes. If $N = 2n$ there are 2 orbits ($\Leftrightarrow$ 2 families of rulings); if $N = 2n+1$ there is 1 orbit.
Let $h \in H^2(Q)$ denote the hyperplane class (pulled back from $P(V)$).

Prop ① If $N = 2n$, let $e, f \in H^{2(n-1)}(Q)$ be defined by $e \cap [Q] = [P(E)]$ and $f \cap [Q] = [P(F)]$, where E and F are in opposite families. Then $e + f = h^{n-1}$, and a $\mathbb{Z}$-basis for $H^*(Q)$ is $1, h, \ldots, \underbrace{h^{n-1}}_{2(n-1)}, e, he, \ldots, h^{n-1}e$. In particular

$$\dim H_{2i}(Q)_{\mathbb{Q}} = \begin{cases} 2 & \text{if } i = n-1 \\ 1 & \text{if } i = 0, \ldots, 2(n-1),\ i \neq n-1. \end{cases}$$

② If $N = 2n+1$ let $e \in H^{2n}(Q)$ be: $e \cap [Q] = [P(E)]$. Then $2e = h^n$ and a $\mathbb{Z}$-basis for $H^*(Q)$ is $1, h, \ldots, h^{n-1}, e, \ldots, h^{n-1}e$, so

$\dim H_{2i}(Q)_{\mathbb{Q}} = 1$, $i = 0, \ldots, 2n$.

2. Quadric bundles

Vector bundle $V \longrightarrow X$ $\longleftrightarrow$ principal $GL(N)$-bundle

v.b. + (nondeg.) quadratic form $\longleftrightarrow$ $O(N)$-bundle

v.b. + L-valued quadratic form $\longleftrightarrow$ $GO(N)$-bundle [explanation: next page]

Here $GO(N) = \{ A \in GL(N) \mid (Av, Aw) = \tau(A)(v,w)$, all $v, w \in \mathbb{C}^N$, where $\tau(A)$ is a scalar depending only on $A \}$

Note: $(\det A)^2 = \tau(A)^N$.

Lemma a) $GO(2n+1) \cong \mathbb{C}^* \times SO(2n+1)$ is connected.

b) $GO(2n)$ has 2 components, defined by $\frac{\det A}{\tau(A)^n} = 1$ (resp. -1).

Until further notice let $V \longrightarrow X$ denote a v.b. with L-valued nondeg quadratic form.

Focus on even rank case: $N = 2n$. Say $V \longrightarrow X$ is orientable if the structure group reduces to the id. component G of $GO(2n)$.

Lemma. There exists a double cover $\pi : \tilde{X} \longrightarrow X$ such that $\pi^* V$ is orientable.

Pf. If V is associated to the $GO(2n)$-principal bundle $E \to X$, let $\tilde{X} = E/G$. $\square$

Remark: A more direct construction of $\tilde{X}$ is: there is an L^{2n}-valued inner product on the line bundle $\Lambda^{2n} V$:

$(v_1 \wedge \cdots \wedge v_{2n}, w_1 \wedge \cdots \wedge w_{2n}) = \det(v_i, w_j)$

So $\Lambda^{2n} V \otimes L^{-n}$ has a quadratic form (values in $\mathbb{1}$) and $\tilde{X} \simeq$ bundle of unit vectors in $\Lambda^{2n} \otimes L^{-n}$.

Reason v.b. with L-valued quadratic form $\Leftrightarrow$ $GO(N)$-bundle:

($\Rightarrow$)
We can cover X with U_i and trivialize our bundle by local sections $e_1, \dots, e_N$ such that $(e_i, e_j) = 0$ unless $i+j = N+1$. However we can't require that $(e_i, e_{N+1-i}) = 1$ since our quad. form doesn't take values in the trivial bundle. All we can do is require that $(e_i, e_{N+1-i}) = (e_j, e_{N+1-j})$ all i, j.

Then if we have another such trivialization $(f_1, \dots, f_N)$ (say over U_j), we end up with the relation $\dfrac{(e_i, e_{N+1-i})}{(f_i, f_{N+1-i})} = \tau \in \mathbb{C}^*$ and the transition matrix between these 2 trivializations

$\therefore A$ with $\tau(A) = \tau$.

($\Leftarrow$) $EGO(N) \to BGO(N)$ has a v.b. $\mathcal{V} = EGO(N) \times^{GO(N)} \mathbb{C}^N$ and $\mathcal{L} = EGO(N) \otimes \mathbb{C}_\tau$

$$\begin{array}{ccc} \mathcal{V} = EGO(N) \times^{GO(N)} \mathbb{C}^N & \quad & \mathcal{L} = EGO(N) \otimes \mathbb{C}_\tau \\ \downarrow & & \downarrow \\ BGO(N) & & BGO(N) \end{array}$$

$\mathcal{V}$ has an $\mathcal{L}$-valued quadratic form. Given a principal $GO(N)$-bundle on X, pull back $\mathcal{V}, \mathcal{L}$ to X via the classifying map $X \to BGO(N)$

3. Characteristic classes and quadric bundles

Let $V \to X$ be a v.b. of rank $2n$ with quadratic form in the trivial line bundle.

Theorem 1. If E and F are max. iso. subbundles of V, then $c_n(E) = \pm c_n(F)$

Define an Euler class $x \in H^*(X, \mathbb{Z}[\frac{1}{2}])$ to be a class such that for any $f: Y \to X$ and any maximal isotropic subbundle $E \subset f^*V$, we have $f^*x = \pm c_n(E)$.

Note that Euler classes are unique (up to sign).

Theorem 2. Given orientable, $V \to X$, there are characteristic classes $\pm y \in H^*(X)$ such that $x = \frac{y}{2^{n-1}}$ is an Euler class in $H^*(X, \mathbb{Z}[\frac{1}{2}])$.

Given orientable $V \to X$, consider

$$\begin{tikzcd} Q \arrow[r, phantom, "\subset"] \arrow[dr, "p"'] & P(V) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

The tautological subbundle is S.

Let $V_{n-1} = S^\perp/S \to Q$. Then V_{n-1} is a vector bundle with quadratic form.

Theorem 3 Let $x_{n-1} \in H^*(Q, \mathbb{Z}[\frac{1}{2}])$ be an Euler class. Then the restriction of x_{n-1} to a fiber gives $\pm(e-f) \in H^*(Q_x)$, i.e., the difference of the ruling classes.

Proofs of theorems Because the bundle $V \to X$ is pulled back from $BSO(2n)$, which is modelled by a smooth f.d. compact manifold, we may assume X is smooth and compact, and so identify homology with cohomology. I'll prove th. 1 assuming E and F are in the same family. Let $H \in H^2(P(V))$, $h =$ pullback to Q, $e = [P(E)]$, $f = [P(F)]$. Then $1, h, \ldots, \underset{e}{h^{n-1}}, h^n, \ldots, h^{2(n-1)}$ are a basis for $H^*(Q)$ over $H^*(X)$. Because e and f

agree on fibers, $e - f = a_{n-1} + h a_{n-2} + \dots + h^{n-2} a_1$ $\quad (a_i \in H^* X)$, so

$i_*(e-f) = 2Ha_{n-1} + 2H^2 a_{n-2} + \dots + 2H^{n-1} a_1$. On the other hand, by std. int theory

$$i_*(e-f) = \sum_{i=0}^{n} H^i \left(c_{n-i}(V/E) - c_{n-i}(V/F)\right). \quad \text{Hence } \underset{(-1)^n c_n(E)}{\underset{\|}{c_n(V/E)}} = \underset{(-1)^n c_n(F)}{\underset{\|}{c_n(V/F)}},$$

proving Thm 1 (in the case where E, F are in the same family)

Now to prove Thm 2, form the tower of quadric bundles $\quad (Q = Q_{n-1})$

$$\begin{tikzcd} V_1 \arrow[d] & & & \\ Q_1 \arrow[r] & P(V_2) \arrow[d] & \cdots & \\ & \vdots & Q_{n-2} \arrow[r] & P(V_{n-1}) \arrow[d] \\ & & & Q_{n-1} \arrow[r] & P(V_n) \arrow[d] \\ & & & & X \end{tikzcd} \qquad \begin{tikzcd} Q_1 = Fl(V_n) \arrow[d] \\ X \end{tikzcd}$$

Now, V_1 is a rank 2 orientable v.b. $\Rightarrow V_1 = E_1 \oplus F_1$ (direct sum of isotropic line bundles).

Therefore each of the bundles $V_2, \dots, V_n$ has 2 max iso subbundles when pulled back to Q_1; E_i and F_i

defined inductively:

$$\begin{tikzcd} S_i^\perp \arrow[r] & S_i^\perp / S_i = V_{i-1} \\ E_i, F_i \arrow[u, phantom, "\subset"] & E_{i-1}, F_{i-1} \arrow[u, phantom, "\cup"] \end{tikzcd} \qquad \Rightarrow c_i(E_i) = h_i c_{i-1}(E_{i-1})$$

Then $c_n(E_n) = -c_n(F_n)$.

Let $f: Fl(V_n) \to X$ be the projection

Let $s = h_2^2 h_3^4 \cdots h_n^{2n-2}$ (pulled back to $Fl(V_n)$). Then $f_*(s \cdot f^* x) = 2^{n-1} x$ for any $x \in H^*(X)$. In particular f^* is injective with $\mathbb{Z}[\frac{1}{2}]$ coeffs.

Define $y \in H^*(X)$ by $y = f_*(s \cdot c_n(E_n))$; then one can show $f^* y = 2^{n-1} c_n(E_n)$ so it is an Euler class. (Reason: consider the commutative diagram

$$\begin{tikzcd} Fl(V) \times_X Fl(V) \arrow[r, "pr_2"] \arrow[d, "pr_1"] & Fl(V) \arrow[d, "f"] \\ Fl(V) \arrow[r, "f"] & X \end{tikzcd}$$

Observe that $pr_1^* c_n(E) = pr_2^* c_n(E)$. This follows since $pr_1^* E$ and $pr_2^* E$ are isotropic subbundles of the pullback of V to $Fl(V) \times_X Fl(V)$ which are in the same family since the bundles are the same on the diagonal. Thus

$$\begin{aligned} f^* f_* (s \cdot c_n(E_n)) &= pr_{1*} pr_2^* (s \cdot c_n(E_n)) \\ &= pr_{1*} (pr_2^* s \cdot p_1^* c_n(E_n)) \\ &= (pr_{1*} pr_2^* s) \cdot c_n(E_n) \\ &= f^* f_* s \cdot c_n(E_n) \\ &= 2^{n-1} c_n(E_n) \qquad \text{as } f_* s = 2^{n-1} \text{ and } f^* 2^{n-1} = 2^{n-1}. \;) \end{aligned}$$

Finally to prove Theorem 3 we can assume X = point. The group $O(2n)$ acts on the tower and if $g \in O(2n)$ not in the identity component, then $g E_1 = F_1$, so $g^* c_1(E_1) = -c_1(E_1)$ since each h_i is invariant we see $g^* c_{n-1}(E_{n-1}) = -c_{n-1}(E_{n-1})$ so $g^* x_{n-1} = -x_{n-1}$. Since $g^* e = f$ and $x_{n-1} = ae + bf$, we conclude that $x_{n-1} = a(e-f)$

Then

$$x_{n-1}^2 = (-1)^n c_{2n-2}(V_{n-1}) = (-1)^n \cdot 2 \qquad \left(\text{use: } c(V_{n-1}) = \frac{c(V_n)}{1-h_n^2} \quad \frac{1}{1-h_n^2} \right)$$

$$\| $$

$$a^2 (e-f)^2 = 2a^2$$

Therefore $a = \pm 1$. $\square$

(quad. form values in $\mathbb{Z}$)

Corollary If $V \longrightarrow X$ is orientable then there exist classes in $H^*(Q;\mathbb{Q})$ restricting to a basis of $H^*(Q_x,\mathbb{Q})$ all $x \in X$. Hence $H^*(Q)_{\mathbb{Q}} = H^*(X)_{\mathbb{Q}} \otimes H^*(Q_x)_{\mathbb{Q}}$

(ii) Can get $y \in A^*X$ (Chow coh.) and R. Field has shown that the exponent can't be improved.

Remarks ① If the quadratic form is L-valued then the corollary still holds (assuming again the bundle is orientable). In fact, this is a general fact about partial flag bundles with connected structure group (goes back to Leray).

② One can ask how Theorems 1-3 generalize to the case where the quadratic form is L-valued. Everything should basically be ok but you might need a higher power of 2 in the denominator, maybe $2^{2(n-1)}$ (I haven't completely checked the details) ... that you don't quite get $c_n(E) = \pm c_n(F)$. In integral cohomology you get something like ... let $z = c_1(L)$, then let

$$\eta(E) = 2^{n-1}\left(c_n(E^\vee \otimes L) - \frac{z}{2} c_{n-1}(E^\vee \otimes L) + \dots \pm \frac{z^{n-1}}{2^{n-1}} c_1(E^\vee \otimes L) \right)$$

Then if E,F are in the same family, $\eta(E) = \eta(F)$, and if E,F are in different families, you get something like

$$\eta(F) = -\eta(F) - (-1)^n z^n \quad (?)$$

and you should be able to get a class y such that $(f^* y)/2^{n-1}$ pulls back to $\eta(E)$

III Proof of deg. locus thm:

Recall the statement: $V \to X$ rank N, X projective, dim d.

Assume V has a quadratic form of constant rank $r = 2n$, values in L, s.t. $S^2(V^*) \otimes L$ is ample

Then $d \leq N - r$

Pf. Consider the even case ($r = 2n$) first. Replace X by orientation double cover $\Rightarrow$ V is orientable. Assume X irreducible, so (let $b_i = \dim H_i(Q) \otimes \mathbb{Q}$) $b_i = 0$ ($i > 2d$), $b_{2d} = 1$.

There is an exact sequence

$$0 \to K \to V \to W \to 0$$

of v.b. on X (K = radical of quadratic form); $\operatorname{rk} K = N - r$, $\operatorname{rk} W = r$ (as vector bundles)

$\tilde{Q} \subset P(V)$, $Q \subset P(W)$ quadric bundles.

not the type we've been considering

Now $\tilde{Q}$ is the 0-scheme of a section of the ample line bundle $\mathcal{O}_{P(V)}(2) \otimes \pi^* L$ so $P(V) \setminus \tilde{Q}$ is affine, hence has the homotopy type of a CW cplx of real dim at most $\dim_{\mathbb{C}} P(V) = N + d - 1$ ([GM p.23]). Since $P(V) \setminus \tilde{Q} \to P(W) \setminus Q$ is a vector bundle with fibers isom. to $\mathbb{C}^{N-r}$ so $P(V) \setminus \tilde{Q}$ is homotopic to $P(W) \setminus Q$.

Conclude:

$$H_j(P(W) \setminus Q) = 0 \quad \text{for } j > N + d - 1$$