D. Gaitsgory   Geometric Langlands for GL_n

X curve, Bun_n stack of rank n bundles on X
$D(Bun_n)$ derived category on Bun_n ($\mathcal{D}$-mods or ℓ-adic sheaves or coeffs in char $> n$)

<u>Geometric Langlands</u>: E local system rank n on X
$\rightsquigarrow$ can attach $\mathcal{F}_E \in Perv(Bun_n) \subset D^b(Bun_n)$
perverse sheaf, characterized by Hecke properties:

$H : D(Bun_n) \longrightarrow D(X \times Bun_n)$ defined by correspondence

Hecke stack

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[d, "s"] & \\ Bun_n & X & Bun_n \end{tikzcd}$$

$= \{(x, \mathcal{M}, \mathcal{M}'; \beta : \mathcal{M} \hookrightarrow \mathcal{M}')$ injection of coherent sheaves, $\mathcal{M}'/\mathcal{M}$ supported at x of length one$\}$
skyscraper

Hecke functor $H(\mathcal{F}) = (\overleftarrow{h} \times s)_!(\overrightarrow{h}^*(\mathcal{F}))[n]$

With parameters: S scheme $\Rightarrow$
$H : D(S \times Bun_n) \longrightarrow D(S \times X \times Bun_n)$

$\Rightarrow$ get iteration $H^{\boxtimes d} : D(Bun_n) \longrightarrow D(X^d \times Bun_n)$

<u>Lemma</u> $H^{\boxtimes d}$ maps to equivariant derived category $D^{\Sigma_d}(X^d \times Bun_n)$

E local system on X
<u>Def</u> An object $\mathcal{F}_E \in D(Bun_n)$ is called <u>Hecke eigensheaf</u> wrt E if we're given an isomorphism
$$H(\mathcal{F}_E) \xrightarrow{\sim} E[1] \boxtimes \mathcal{F}_E \quad \text{s.t. for all } d$$
(enough in fact to take $d=1$)
$$H^{\boxtimes d}(\mathcal{F}_E) \xrightarrow{\sim} \underbrace{E[1] \boxtimes \cdots \boxtimes E[1]}_{d \text{ times}} \boxtimes \mathcal{F}_E$$
respects Σ_d-equivariant structures on both sides

<u>Easy</u>: if E has wrong rank ($\neq n$) then no automorphic sheaves exist

Conjecture: If rk $E = n$ and E is irreducible, $\mathcal{F}_E$ exists
(Hecke eigensheaves, $\neq 0$, exist)

[Uniqueness hasn't been established]

Frenkel-Vilonen-Gaitsgory following Laumon following Drinfeld: reduce to a vanishing conjecture...

Good behavior of Rankin Selberg convolutions of $L(\pi, E)$

$\overset{\Longrightarrow}{\text{converse theorem}}$ existence of automorphic forms

Vanishing conjecture (analogue of Rankin-Selberg...) $\xrightarrow[FGV]{}$ $\exists$ automorphic sheaves

Take E local system (not nec. rank n) $\Rightarrow$ define averaging functor

$$Av_E^d : D(Bun_n) \longrightarrow D(Bun_n)$$

$$\begin{tikzcd} & Mod_n^d \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ Bun_n & & Bun_n \end{tikzcd}$$

= stack classifying triples $(M, M'; \beta. M \hookrightarrow M')$ with length $(M'/M) = d$.

$(d=1 \Rightarrow \mathcal{H}\ell)$

$$\begin{tikzcd} Mod_n^d \arrow[d, "s"'] & \overset{\circ}{Mod}_n^d \arrow[l, "j"'] \arrow[d, "\overset{\circ}{s}"] \\ X^{(d)} & \overset{\circ}{X}^{(d)} \arrow[l, hook'] \end{tikzcd}$$

d distinct points

Laumon: $E \longmapsto \mathcal{L}_E^d \in Perv(Mod_n^d)$

via $E \mapsto E^{(d)}$ on $X^{(d)}$

$\mapsto \overset{\circ}{E}^{(d)}$ on $\overset{\circ}{X}^{(d)}$ restriction

Now take $\overset{\circ}{E}^{(d)}$ & pullback to $\overset{\circ}{Mod}_n^d$

& take intermediate extension $\boxed{\mathcal{L}_E^d = j_{!*} \overset{\circ}{s}^*(\overset{\circ}{E}^{(d)})}$

$(\neq s^* E^{(d)}\ !)$

Def : $Av_E^d(F) = \overleftarrow{h}_! (\overrightarrow{h}^*(F) \otimes \mathcal{L}_E^d)[nd]$

... integrate X out of previous picture H by cutting local system E along it.

<u>Conjecture</u> Suppose $rk\, E > n$, E irreducible & $d > (2g-2)\, rk\, E \cdot n$

Then $Av_E^d = 0$

FGV: This conjecture implies Langlands conjecture

Over $\mathbb{F}_q$ this follows from Lafforgue..
want independent proof.

<u>Theorem</u> Under some conditions & assuming vanishing conjecture for all smaller rk of E $\Longrightarrow$ the functor $Av_E^d : D(Bun_n) \circlearrowleft$ is <u>exact</u> (so sends perverse $\circlearrowleft$).

<u>Claim</u> Exactness theorem implies vanishing conjecture
Explanation:
(1) Suffices to show $Av_E^d(\mathcal{F}) = 0$ for all $\mathcal{F}$ perverse by exactness
(2) " " " (Pointwise) Euler characteristics of stalks are zero for all points $M \in Bun_n$
(Since there's a part where it's a local system in some dimension, so χ = the dimension!)

(3) Euler chars $\chi_M(Av_E^d(\mathcal{F}))$ depend <u>only</u> on $\operatorname{rk} E$, not on E itself!
- follows from Deligne's theorem, can compare étale locally isomorphic E.

(4) We know at least one E' for which vanishing holds!
$\Rightarrow$ we're done.
The good E' we know: enough to do eg for $\overline{\mathbb{Q}}_\ell$ coeffs over <u>finite fields</u> use <u>cyclic covers</u>....

Proof of exactness theorem: replace Av^d by the iteration of Av^1:

$$\widetilde{Av}_E^d = \underbrace{Av_E^1 \circ \dots \circ Av_E^1}_{d \text{ times}}$$

- reinterpret as composition

$$D(Bun_n) \xrightarrow{H^{\boxtimes d}} D(X^d \times Bun_n)$$

$$X^d \xleftarrow{q} X^d \times Bun_n \xrightarrow{p} Bun_n$$

$$\widetilde{Av}_E^d(\mathcal{F}) = p_!\left(H^{\boxtimes d}(\mathcal{F}) \otimes q^*(E^{\boxtimes d})\right)$$

Corollary $\widetilde{Av}_E^d : D(Bun_n) \longrightarrow D^{\Sigma_d}(Bun_n)$ get sheaves with Σ_d action

Lemma $Av_E^d(F) \simeq (\hat{Av}_E^d(F))^{\Sigma_d} \simeq (\tilde{Av}_E^d(F))_{\Sigma_d}$ $\Rightarrow$ exact!

(left exact: invariants; right exact: coinvariants)

(char $\neq 0$ ~~don't~~ need to do this in derived sense & invts, coinvts not apriori zero).

Problem: $\hat{Av}$ not exact in general! – due to nonexactness of Hecke functor.

Toy case $n=1$: Av_E^1 is exact:

$$\begin{tikzcd} Pic & X \times Pic \arrow[l, "m"'] \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & Pic \end{tikzcd}$$

$$Av_E^1(F) = p_!(q^*(E) \otimes m^*(F))[1]$$

Problem is with pushforward, can get cohomologies in degree ± 1. So assume F is an irred perverse sheaf on Pic s.t. $p_!(q^*E \otimes m^*F)[1]$ has nonzero H^1.

$\Rightarrow$ by adjunctions get map $m^*(F) \xrightarrow{\neq 0} E^* \boxtimes F'$ some F'

- taking F' irred WLOG, since m^*F irred

$\Rightarrow$ this map is isomorphism:

but can't have anything like Hecke eigensheaf of wrong rank!

Why? apply functor again

$$Pic \xleftarrow{m} X \times Pic \xleftarrow{id \times m} X \times X \times Pic$$

$(id \times m)^* \circ m^*(F) \in Perv(X \times X \times Pic)$ is at the same time Σ_2-equivariant & of the form $E^* \boxtimes m^*(F')$

$$\Rightarrow (id \times m)^* m^* F \simeq E^* \boxtimes E^* \boxtimes F''$$

Σ_2 (swapping the two E^*), $\times$ Σ_2 acting on F''

Get contradiction: look at functor $F \longrightarrow ((id \times m)^* \circ m^*(F))|_{\Delta \times Pic} \otimes Sign)^{\Sigma_2}$

On one hand the functor $=0$ by commutativity of Pic,

on other hand get $Sym^2 E^* \otimes (F'')^{sign} \oplus \Lambda^2 E^* \otimes (F'')^{triv} \neq 0$

since rk $E \neq 1$

This proof would work in general if H were exact $\Longrightarrow$ make it exact artificially!!

For any base S introduce quotient categories

$D(S \times Bun_n) \longrightarrow \tilde{D}(S \times Bun_n)$

Note: If C is triangulated category, $C' \subset C$ triangulated sub $\Rightarrow$ construct C/C'. Can do all this compatibly with t-structures as well.

We'll do this compatibly with all six operations

$S_1 \longrightarrow S_2 \Rightarrow D(S_1 \times Bun_n) \rightleftarrows D(S_2 \times Bun_n)$

— quotient out by things that are degenerate in sense of Whittaker model, & stuff supported on very unstable bundles.

<u>Property 1 of $\tilde{D}$</u> $\quad H : D(S \times Bun_n) \longrightarrow D(S \times X \times Bun_n)$ descends to $\tilde{D}$'s, where it is <u>exact</u>.

[H is exact on Whittaker model!]

$\dashrightarrow$ get Av_E', $\tilde{Av}_E^d$, Av_E^d all defined on $\tilde{D}(Bun_n)$ & are exact using $n=1$ proof.

- so get exactness modulo some subcategory.

<u>but this is enough</u>: introduce cuspidal category $D_{cusp}(Bun_n) \subset D(Bun_n)$

<u>Property 2</u> $\quad Hom(D_{cusp}(Bun_n), Ker(D \to \tilde{D})) = 0$

<u>Proof of exactness</u> $\quad \forall \mathcal{F}, Av_E^d(\mathcal{F}) \in D_{cusp}(Bun_n)$ by inductive hypothesis.

So for $\mathcal{F}$ perverse look at truncation

$D_{cusp} \ni Av_E^d(\mathcal{F}) \longrightarrow \tau^{>0}(Av_E^d(\mathcal{F})) \in Ker(D \to \tilde{D})$,

can't be zero unless $RHS = 0$. But in <u>quotient</u> category RHS is zero and so by property 2 the Hom is zero $\Longrightarrow$ RHS is zero!

[Average is cuspidal since Av commutes with constant term functor]