A. Beilinson - On Langlands Correspondence in the deRham setting I 3/18/02

Local Picture today. de Rham version: differs from ℓ-adic setting, use methods unavailable ℓ-adically..

Usual Langlands: relates two seemingly unrelated objects.
G split reductive gp. F local field $= k((t))$ for us
(usual Langlands: k finite)
Rep theory: G(F) (locally compact gp) & reps of G
Galois theory: $G^\vee$ Langlands dual: dual root data to G
(- consider $/\mathbb{Q}$ or $/\mathbb{C}$ or $/\mathbb{Q}_\ell$.)
Reps $\mathrm{Gal}(\bar{F}/F) \longrightarrow G^\vee \Longleftrightarrow$
$G^\vee$ local systems on Spec F (ℓ-adic)

Expect decomposition of reps of G(F) into series labelled by Galois data.
More precise: Bernstein center = End of identity functor of rep category, which is fibered over Spec of the Bernstein center

$\Rightarrow$ expect Spec(Bernstein center) = set of $G^\vee$-local systems

Principal methods are *global* - no direct purely local relation.

deRham version: k fixed field of char. 0 (eg $\mathbb{C}$)
$G^\vee$-loc systems: now in de Rham sense
$= G^\vee$ bundles with connection on Spec F.
- depend on *continuous parameters* ...
formal differential eqns (no *Stokes* parameters), arbitrary irregular singularities allowed!

Rep theory side: $\mathfrak{g}$ f.d reductive Lie algebra
$\mathfrak{g}(F)$ ∞-dim (topological) Lie algebra..
better consider reps of Kac-Moody central extension, at *level* κ = Ad-invt quadratic form on $\mathfrak{g}$: cocycle
$a, b \longmapsto \mathrm{Res}\, \kappa(b, da)$ giving central extension

Note: everything here will be purely local - advantage of deRham setting

(1 will act as identity for our reps)

Rep theory depends on κ.. should consider special κ : integral & negative in strong sense (less than critical). & nondegenerate (e.g. $\mathfrak{g}$ torus: nondegen integral scalar product on corresponding lattice)

Format of conjecture : (rough)

$\mathcal{LS}$ = moduli of $G^\vee$-local systems on Spec F ~ stack over k, not algebraic (just know what families mean)

ⓐ Want to define an associative topological algebra (over $\mathcal{O}_{\mathcal{LS}}$) $\mathcal{A}$ on $\mathcal{LS}$ together with map of Lie algebras $\mathfrak{g}(F)^\kappa \longrightarrow \mathcal{A}$

Given module over $\mathcal{A}$ $\longleftrightarrow$ "quasicoherent" sheaf on $\mathcal{LS}$, its global sections carry action of $\mathfrak{g}(F)^\kappa$

$\Rightarrow$ functor $\mathcal{A}$-modules $\xrightarrow{\Gamma}$ $\mathfrak{g}(F)^\kappa$-modules

Want this to be an equivalence of categories .

... so modules over $\mathfrak{g}(F)^\kappa$ decomposed wrt "spectral parameters" $\mathcal{LS}$.

Very natural construction - (e.g. wrt $\mathfrak{g}(F)$-action)

ⓑ Given local system $\in \mathcal{LS}$ can ask which $\mathfrak{g}(F)^\kappa$-modules are supported here? want explicit geometric description - at least for tame local systems (regular sings)

Comment a. will come from natural vertex algebra assoc to G -- e.g. torus $\Rightarrow$ lattice Heisenberg vertex algebra.

- algebra will be equipped with $G^\vee$-action ... so can twist vertex algebra by any $G^\vee$-local system, & these are fibers of $\mathcal{A}$!
- can't do on level of plain associative algebras (twisting by $\mathcal{LS}$)

- If moduli of LS happens to be an affine variety this would be same as map $\mathbb{C}[LS] \to$ Center of $\mathfrak{g}(F)^{\kappa}$ --- but both sides $\mathbb{C}[LS]$, center are trivial! naive version doesn't work.

Def of vertex algebra analogous to getting lattice Heisenberg from plain enveloping algebra - add extra generators.

Center is somehow internal usually - but here construct "external center" .. text with carcass outside ....

Part ⓑ (for tame local systems with unipotent monodromy)

$G(F)$ is an ind-scheme from POV of k (inductive limit of affine ~~ga~~ schemes)

$$\begin{tikzcd} G(F) \arrow[r, phantom, "\supset"] & G(\mathcal{O}) \arrow[r, phantom, "\supset"] \arrow[d, "\text{group scheme}"'] & \text{Iwahori} \arrow[d] \\ & G \arrow[r, phantom, "\supset"] & B \end{tikzcd}$$

$\phi = G(F)/\text{Iwahori}$: ind-*proper* ind scheme : *Affine Flag space*

$\Rightarrow$ category $\mathcal{M}(\phi)$ of $\mathcal{D}$-modules on ϕ: [*right* $\mathcal{D}$-mod] union of f.d varieties wrt closed embeddings, so look at union of $\mathcal{D}$-submodules supported on f.d. piece.

--- *right* $\mathcal{D}$-mods make sense as *sheaves* here: these embed into ~~pushforward~~ each other to glue unlike left (need to twist).

$$\Gamma : \mathcal{M}(\phi) \longrightarrow \mathfrak{g}(F)\text{-modules}$$

- really should twist by appropriate line bundle! κ defines central extension of $G(F)$ by $\mathbb{G}_m$ $\to G(F)^{\kappa}$, look at equivariant line bundles for $G(F)^{\kappa}$ on ϕ: they form a torsor over weight lattice of G (*affine* condition: 1 acts by 1) - carries action of affine Weyl fin many orbits

Pick ample line bundle $\mathcal{L}$ from any W^{aff}-orbit
(many ways possible)

$\Rightarrow$ $\Gamma_{\mathcal{L}} : M \longmapsto \Gamma(\phi, M \otimes \mathcal{L})$ $\mathfrak{g}(F)^{K}$-module
(exact fully faithful functor)

Wish: $\Gamma_{\mathcal{L}}$ produces equivalence of categories

$$\prod_{\mathcal{L} \in W^{aff}\text{-orbits}} \mathcal{M}(\phi) \xrightarrow[\Gamma_{\mathcal{L}}]{\sim} \mathfrak{g}(F)^{K}\text{-modules supported at the nilpotent local systems}$$

- eg all category $\mathcal{O}$, Vermas etc come this way (from tame local systems). Functors $\Gamma_{\mathcal{L}}$ depend on choice of $\mathcal{L}$ but have geometric intertwiners

Case $G = T$ torus

[A = lattice Heisenberg & its twists by local systems
Rep theory side: reps of Lie Heisenberg algebra
- decompose its reps wrt reps of all twisted lattice Heisenberg algebras !]

Very brief introduction to vertex algebras :

Work over a curve X (eventually look a disc)
A = quasicoherent $\mathcal{O}_X$-module

Def: "Factorization structure" on A = a collection of $\mathcal{O}_{X^n}$-modules all n $\{A_{X^n}\}$ with compatibility data

Informally: $A_X = A$, key property: $\forall (x_1, \dots x_n) \in X^n$ consider fiber $A_{(x_1, \dots x_n)}$, demand that it equals $\bigotimes_{x \in \{x_1 \dots x_n\}} A_x$ where we consider $(x_1, \dots x_n)$ as plain subset of X - no multiplicities (one copy for each distinct point).

Precisely on X^2 A_{X^2}

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times X \\ & U = X \times X \setminus \Delta \arrow[u, hook, "j"'] \end{tikzcd}$$

demand $\Delta^* A_{X^2} = A$
$j^* A_{X^2} \cong j^*(A \boxtimes A)$
plus action of switching factors compatible with $A \boxtimes A$,

Structure is completely local: gluing of $A \boxtimes A$ off Δ to A on Δ.

<u>Def</u> A chiral algebra structure on A is a factorization structure s.t. 1. all A_{X^n} flat in transversal direction to diagonal 2. A has a unit: global section $1 \in \Gamma(X, A)$ s.t. $\forall a \in A$, $a \boxtimes 1$ off Δ extends to diagonal: $a \boxtimes 1 \in A_{X^2} \subset j_* j^* A \boxtimes A$ & $\Delta^*(a \boxtimes 1) = a$

Note: such structure yields canonically a $\mathcal{D}$-mod structure on A:

$$\begin{tikzcd} p_1^* A_X \arrow[r, hook] & A_{X^2} & p_2^* A_X \arrow[l, hook'] \\ a \arrow[r, mapsto] & a \boxtimes 1 \quad 1 \boxtimes a & a \arrow[l, mapsto] \end{tikzcd}$$

.. pulled back to diagonal these maps are <u>isomorphisms</u> - so since our objects are <u>flat</u> transversally to diagonal get isomorphism on formal nbhd of diagonal $\Longleftrightarrow$ $\mathcal{D}$-module structure

Operator Product Expansion:

$$\begin{tikzcd} & A_X \boxtimes A_X \arrow[r, hook] \arrow[dl, "{[,]}"'] \arrow[d, "OPE", "a_1 \boxtimes a_2 \mapsto a_1 \circ a_2"'] & j_* j^* A_X \boxtimes A_X = j_* j^* A_{X^2} \arrow[d] \\ \Delta_* A_X & A_X^{(n)}((t_1 - t_2)) \arrow[l] \arrow[r, equal] & j_* j^* \hat{A}_{X^2} \end{tikzcd}$$

in local parameter

complete along diagonal & localize wrt equation of Δ

Algebraic part: take only polar part

OPE completely determines A_{X^2} hence everything! just gluing data.