

A. Beilinson - On Langlands Correspondence
in the de Rham setting I

3/18/02

Local Picture today. de Rham version: differs from basic setting, use methods unavailable radically...

Visual Langlands: relates two seemingly unrelated objects.
 G split reductive gp, F local field = $k((t))$ for us
 (usual Langlands: k finite)
 ↑ Rep theory: $G(F)$ (locally compact gp) & reps of G
 ↓ Galois theory: G^V Langlands dual: dual root data to G
 (- consider \mathbb{Q} or \mathbb{C} or \mathbb{R} ...)
 Repts $G(F) \rightarrow G^V \iff$
 G^V local systems on $\text{Spec } F$ (adic)

Expect decomposition of reps of $G(F)$ into series
labelled by Galois data

More precise: Bernstein center = End of idempotent
functor of rep category, which is fibred over
 Spec of the Bernstein center

\Rightarrow expect $\text{Spec}(\text{Bernstein center}) = \text{Set of } G^V\text{-local systems}$

Principal methods are global - no direct purely local
relation.

de Rham version: k fixed field of char. 0 (eg \mathbb{C})

G^V -loc systems: now in de Rham sense
= G^V bundles with connection on $\text{Spec } F$.

- depend on continuous parameters ...
formal differential eqns (no Stokes parameters),
arbitrary irregular singularities allowed!

Rep theory side: \mathfrak{g} fid reductive Lie algebra
 $\mathfrak{g}(F)$ ∞ -dim (topological) Lie algebra...
better consider reps of Kac-Moody central
extension, at level $\kappa = \text{Ad-invt quadratic}$
form on \mathfrak{g} : cocycle
 $a, b \mapsto \text{Res}(b, da)$ giving central extension

Note: everything here will be purely local - advantage of deformation

($\mathbb{1}$ will act as identity for our reps)

Rep theory depends on k ... should consider

Special k : integral & negative in strong sense (less than critical) & nondiscrete

(e.g. of torus: nondegen integral scalar product on corresponding lattice)

Format of conjecture: [rough]

$\mathcal{L}\mathcal{S}$ = moduli of G^v -local systems ^{on $\text{Spec } F$} - start over k ,
not algebraic (just know what families near)

(a) Want to define an associative topological algebra (or $G_{\mathcal{L}\mathcal{S}}$)
 \mathcal{A} on $\mathcal{L}\mathcal{S}$ together with map of Lie algebras
 $\mathfrak{g}(F)^k \rightarrow \mathcal{A}$

Given module over $\mathcal{A} \iff$ "quasicoherent" sheaf
on $\mathcal{L}\mathcal{S}$, its global sections carry action of $\mathfrak{g}(F)^k$

\implies functor \mathcal{A} -modules $\xrightarrow{\Gamma} \mathfrak{g}(F)^k$ -modules
Want this to be an equivalence of categories.

... so modules over $\mathfrak{g}(F)^k$ decomposed w.r.t
"spectral parameters" $\mathcal{L}\mathcal{S}$.

Very natural construction (e.g. w.r.t $\mathfrak{g}(F)$ -action)

(b) Given local system ^{eff} can ask which $\mathfrak{g}(F)^k$ -modules
are supported here? want explicit geometric
description - at least for tame local systems (regular sing.)

Comment a. will come from natural vertex algebra
assoc. to G -- e.g. torus \rightarrow lattice Heisenberg
vertex algebra.

- algebra will be equipped with G^v -action...
so can twist vertex algebra by any G^v -local
system, & these are fibers of \mathcal{A} !

- can't do on level of plain associative algebras
(twisting by $\mathcal{L}\mathcal{S}$)

• If moduli of IS happens to be an affine variety
 this would be same as map $C[[\mathbb{L}S]] \rightarrow \text{center of } \mathfrak{g}(F)^k$
 --- but both sides $C[[\mathbb{L}S]]$, center
 are trivial! naive version doesn't work.

Def of vertex algebra analogous to getting lattice Heisenberg
 from plain enveloping algebra - add extra generators.

Center is somehow internal usually - but have construct
 "external center" ... that with cocycles outside

Part (B) (for tame local systems with unipotent monodromy)

$G(F)$ is an ind-scheme from POV of k (inductive
 $G(F) \supset G(O) \supset \text{Iwahori}$ (limit of affine
 group schemes))
 group scheme \downarrow
 $G \supset B$

$\phi = G(F)/\text{Iwahori}$: ind-proper ind scheme : Affine flag space

\Rightarrow category $\mathcal{M}(\phi)$ of \mathbb{D} -modules on ϕ : [right] \mathbb{D} -mod!
 union of f.d. varieties with closed embeddings,
 so look at union of \mathbb{D} -submodules supported
 on f.d. piece.

--- right \mathbb{D} -mods make sense as sheaves
 here : these embed into \mathbb{P}^1 and glue each other to
 give unlike left (need to twist).

$\Gamma : \mathcal{M}(\phi) \longrightarrow \mathfrak{g}(F)$ -modules

- really should twist by appropriate line bundle!
 K defines central extension of $G(F)$ by \mathbb{G}_m
 $\rightarrow G(F)^k$, look at equivariant line bundles
 for $G(F)^k$ on ϕ : they form a torsor
 over weight lattice of G (affine condition:
 1 acts by 1) - carries action of affine Weyl
 for many orbits

Pick ample line bundle \mathcal{L} from any W^{aff} -orbit
(many ways possible)

$$\Rightarrow \Gamma_{\mathcal{L}} : M \longmapsto \Gamma(\phi, M \otimes \mathcal{L}) \quad \text{exact fully faithful functor}$$

$\mathfrak{g}(\mathcal{F})^K$ -module

Wish: $\Gamma_{\mathcal{L}}$ produces equivalence of categories

$\prod_{W^{aff}\text{-orbits}} \mathcal{M}(\phi) \xrightarrow{\Gamma_{\mathcal{L}}} \mathfrak{g}(\mathcal{F})^K$ -modules supported on the nilpotent local systems

- eg all category \mathcal{O} , Verma et cetera way (from the local systems). Functors $\Gamma_{\mathcal{L}}$ depend on choice of \mathcal{L} but have generic intertwiners

Case $G = T$ torus

[$A =$ lattice Heisenberg & its twists by local systems
Rep theory side: reps of $\mathbb{C}c$ Heisenberg algebra
- decompose its reps wrt reps of all twisted lattice Heisenberg algebras!]

Very brief introduction to vertex algebras:

Work over a curve X (eventually look a disc)

$A =$ quasicoherent $\mathbb{C}[c]$ -module

Def: "Factorization structure" on $A =$ a collection of $\mathbb{C}[c]$ -modules all in $\{A_{X_n}\}$ with compatibility between

Intd. $A_X = A$, key property! $\forall (x_1, \dots, x_n) \in X^n$
consider fiber $A_{(x_1, \dots, x_n)}$, demand that it equals $\bigotimes_{i=1}^n A_{x_i}$
where we consider (x_1, \dots, x_n) as plain subset of X - no multiplications (one copy for each distinct point).

$$\begin{array}{ccc} \text{Precisely on } X^2 & A_{X^2} & X \xrightarrow{\Delta} X * X \\ & & \uparrow j \\ & & U = X * X * \Delta \end{array}$$

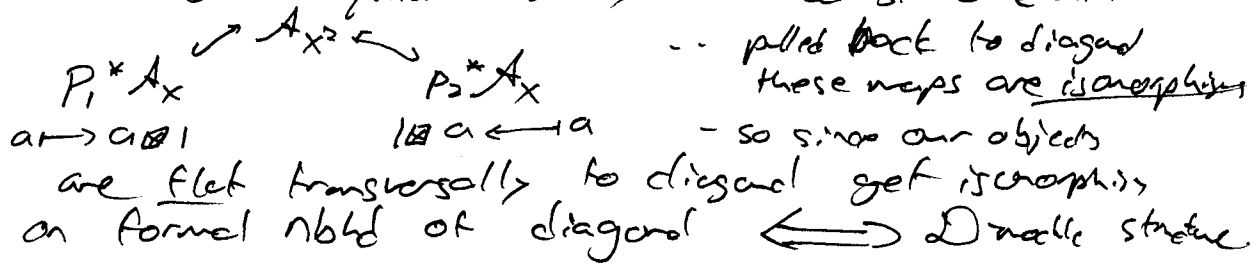
demand $\Delta^* A_{X^2} = A$
 $j^* A_{X^2} = j^*(A \boxtimes A)$

plus action of switching factors compatible with $A \boxtimes A$

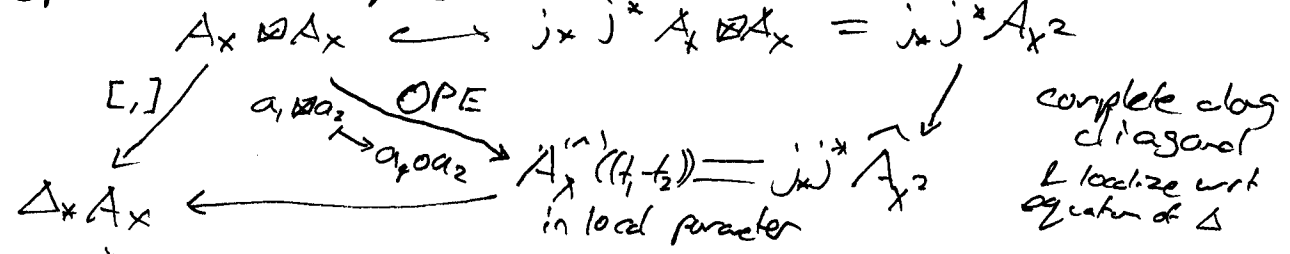
Structure is completely local: study of $A \otimes A$ off Δ to A or Δ .

Def A chiral algebra structure on A is a factorization structure s.t. 1. all A_x flat in transversal direction to diagonal 2. A has a unit: global section $1 \in \Gamma(K, A)$ s.t. $\forall a \in A, a \otimes 1$ off Δ extends to diagonal: $a \otimes 1 \in A_{x^2} \subset \text{inj}^* A \otimes A$
 $R \Delta^*(a \otimes 1) = a$

Note: such structure yields canonically a D-mod structure on A :



Operator Product Expansion:



Algebraic part: take only polar part
 OPE completely determines A_{x^2} hence everything!
 just gluing data.