G. Faltings - Introduction to Shimura Varieties 3/19/02

1. <u>Elliptic Curves</u> $\mathbb{H}/PSL(2,\mathbb{Z})$ = isom classes of elliptic curves / $\mathbb{C}$

- really moduli problem defined over $\mathbb{Q}$ $\Longrightarrow$ algebraic variety defined over $\mathbb{Q}$

Shimura: such spaces tend to be defined over number fields

2. <u>Hermitian symmetric spaces</u> (higher dim):

G/K G semisimple $\supset K$ maximal compact

Hermitian if has Hermitian structure (in particular even dim!) $\Longleftrightarrow$ map $U(1) \to G$.

K = 0-eigenspace, & only others are ± 1 eigenspaces.

Shimura variety roughly $\Gamma \backslash G/K$ Γ arithmetic

A_n type: $U(p,q)/U(p) \times U(q)$ ---> take $U(1)$ in either factor

B type: $\to SO(p,2)/SO(p) \times SO(2)$ ---> our $U(1)$

$\to SO^*(2n)/U(n)$ ---> start from complex vector space

$SO^*(2n) = SO(2n,\mathbb{C}) \cap U(n,n)$

E_6, E_7 types

$Sp \rightsquigarrow \mathbb{H}_g$ Siegel upper half space

Quotient by arithmetic group: usually classifies abelian varieties with Hodge cycles.

Most interesting ones: P E L types

(P = polarization, E = endomorphism, L = level)

e.g. Hilbert modular surfaces: $K = \mathbb{Q}(\sqrt{+d})$ real quadratic field. {A = Abelian variety (2-dim) with action of $\mathcal{O}(K)$ integers + polarization}

$\updownarrow$

quotient of $\mathbb{H} \times \mathbb{H}$ by Γ:

$H_1(A,\mathbb{Z})$ has rank 2 over $\mathcal{O}_K$ (rk 4 over $\mathbb{Z}$)

-- if class number > 1 these are not all isomorphic, classified by determinant $\in Pic(\mathcal{O}_K)$

- so for each fixed determinant get Shimura variety.

Want to use Hecke operators: Hecke correspondence classifies isogenies $A \longrightarrow A'$ of "given type"

"Given type" : eg fix O_K-type of cokernel of map on H_1's
... changes invariant $H_1(A, \mathbb{Z})$, so should consider all isom classes of such to get Hecke action
$\Rightarrow \coprod_{\text{determinant}}$ (moduli space)

Really need adèlization ; instead of $\Gamma \backslash G/K$ take $G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f \times K_\infty$

- G reductive group over $\mathbb{Q}$, $\mathbb{A}$ = adèles = $\mathbb{Q} \otimes \hat{\mathbb{Z}} \times \mathbb{R}$
  K_f = compact open in $G(\mathbb{A}_f)$
  $\mathbb{A}_f$ = finite adèles
  K_∞ = maximal compact in $G(\mathbb{R})$

$G(\mathbb{Q})$ plays role of Γ (discrete) :
flexibility : many reasonable K_f's compact open,
only one reasonable K_∞ (max compact)

Shimura variety : special case $G(\mathbb{R})/K_\infty$ hermitian

Advantage : G(field) easier than G(Dedekind ring) !
K_f hides some things :
$G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f \times K_\infty = \coprod \Gamma \backslash G(\mathbb{R}) / K_\infty$
$\Gamma = G(\mathbb{Q}) \cap$ conjugate of K_f

These are $\mathbb{C}$-points of an algebraic variety defined over a number field, the reflex field E :

at ∞ place have Hermitian symmetric space $\rightsquigarrow$
$U(1) \to G(\mathbb{R})$ with centralizer K_∞
$\Rightarrow \mathbb{G}_m \to G(\mathbb{C})$, conjugacy class of this map is defined over a number field, E = field of definition of this conjugacy class.
.... difficult in general : need to show conjugate of Shimura variety is Shimura variety

Illustration! Shimura curves

E = totally real field, B = Quaternion algebra, indefinite at one ∞ place & definite elsewhere,

$G = B^*$. $G(\mathbb{R}) = GL_2(\mathbb{R}) \times$ compact

$U(1) \to G$ goes into $GL_2(\mathbb{R})$ factor, E is the reflex field here.

$\Rightarrow$ Shimura variety defined over E...

but don't find natural moduli problem over E!

Instead (Shimura): consider $F \supseteq E$ a totally complex quadratic extension.

Consider moduli problem over F: A = abelian variety with multiplication by order in $B \otimes_E F$

$H_1(A, \mathbb{Q}) = B \otimes_E F$ regular representation,

action of F on $\mathrm{Lie}(A)_{\mathbb{C}} = \bigoplus_{E \hookrightarrow \mathbb{R}}$ (E-eigenspaces)

- $E \hookrightarrow \mathbb{R}$ the place where $B_{\mathbb{R}}$ is indefinite, want F to act by a number (4) of copies of $F \otimes_{\mathbb{R}} \mathbb{C}$, both embeddings $F \hookrightarrow \mathbb{C}$ have same multiplicity in here
- $E \hookrightarrow \mathbb{R}$ s.t. $B_{\mathbb{R}}$ is definite! F acts by scalars under $F \hookrightarrow \mathbb{C}$.

Take polarization s.t. Involution on $B \otimes F$ is product of Involutions

$\Rightarrow$ Shimura variety defined over F: curve/F.
Show it descends to E: for ~~isomorphic~~ different F's get isomorphic curves.

$\boxed{\text{Integral models}}$ $\mathcal{O}_E \hookrightarrow E$ integers: would like a smooth model / $\mathcal{O}_E$ ... should hold where K_f has good reduction

eg elliptic curves with level n structure $\Rightarrow$ good reduction over $\mathbb{Z}[\frac{1}{n}]$ ($\leftrightarrow K_f$: 1 mod n)

PEL type: relatively easy since given by moduli problem which has good reduction

Hodge type: (have higher Hodge cycles) sometimes possible
eg moduli of K3 surfaces $SO(19,2)/SO(19)\times SO(2)$
(polarized)
$\Rightarrow$ abelian variety A with $H_1(A)$ = Spin rep of $SO(19,2)$
but ~~not~~ of PEL type — rather Hodge cycles
in degree $(2,2)$ given by invariants of Spin group.

E_6, E_7: ???

Tame bad reduction: eg elliptic curves & $\Gamma_0(p)$:
isogenies $E \to E'$ of degree p.
Over $\mathbb{Z}_p$ have semistable local model, singularity of
type $\{xy = p\}$

$H_1(E) \longrightarrow H_1(E')$ symplectic similitude of multiplier p.

$$\begin{array}{ccc} R^2 & \longrightarrow & R^2 \\ \cup & & \cup \\ R & \longrightarrow & R \end{array}$$ Hodge filtration

On Hodge piece R ($H^0(\Omega^1)$) get multiplication by x, on quotient R get mult by p, & $xy = p$.

- doesn't work for p^2.

Abelian varieties (principally polarized) $A \xrightarrow{p^g} A'$
$\Rightarrow$ matrices $ST = TS = p$, S, T symmetric
matrices — describes local singularities

Drinfeld level structure: E = elliptic curve, $y, x \in E$
is Drinfeld level n structure if $nx = 0$, $ny = 0$
& $E[n] = \sum_{i,j \in \mathbb{Z}/n} (ix + jy)$ as divisors

$\to$ gives regular local models even if $p \mid n$ ....
not semistable though. Can be generalized to $U(p,1)$-groups:
get 1-dim formal groups of height $p+1$
$\Rightarrow$ used in Harris–Taylor

$\boxed{\text{Compactification}}$
a. Baily–Borel–Satake–Shimura (minimal) compactification:

$Sh \hookrightarrow Sh^*$: closure of projective embedding given by modular forms .. normal, but has singularities at boundary. e.g for abelian varieties $A_g^* = A_g \amalg A_{g-1} \amalg \dots \amalg A_0$

- codimension is g ... sometimes can define over integers! For PEL type definable over $\mathcal{O}_E$ with some good properties

b. Toroidal compactifications : [don't behave well under change of level] — resolutions of singularities

... Note Sh_G = pro-scheme of $G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f \times K_\infty$ as K_f varies — projective system... toroidals don't behave nicely for this system.
The projective system has an action of $G(\mathbb{A}_f)$
<-> Hecke operators. [log smooth...]

$\boxed{\text{Counting Points}}$ Good model of $Sh_G \Rightarrow$ describe $Sh_G(\mathbb{F}_q)$:
Langlands-Rapoport conjecture: use Lefschetz trace formula for Frobenius to describe # points (isogeny classes) ... mess!

eg elliptic curve : isogeny classes $\longleftrightarrow$ characteristic polynomial of Frobenius.

Relate to <u>trace formula</u> : describe automorphic forms = functions on $G(\mathbb{Q}) \backslash G(\mathbb{A}) \circlearrowleft G(\mathbb{A})$

Formally: $\varphi \in C_c^\infty(G(\mathbb{A}))$, $\_ * \varphi$ given by a kernel

$$\sum_{\gamma \in G(\mathbb{Q})} \varphi(x^{-1}\gamma y), \quad Tr(\varphi) = \sum_\gamma \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(x^{-1}\gamma x)\, dx$$

$$= \sum_{\gamma \text{ conj. classes}} \int_{Z(\gamma) \backslash G(\mathbb{A})} \varphi(x^{-1}\gamma x)\, dx \longleftarrow \text{Orbital integrals}$$

Noncompact case! need to deal with Eisenstein series.... mess!
Application a: compare with previous mess
b: G, G' groups Want relation between conj classes & orbital integrals
-- Fundamental lemma