M. Rapoport - Local Models of Shimura Varieties
(w. G. Pappas)

Moduli of abelian varieties with parahoric level structure at p ... congruences of modular forms vanishing cycles etc..

Typical example: F = totally real field,
$\tilde{F}/F$ totally imaginary quadratic extension.
$V = \tilde{F}$-vector space with $(\,,\,) : V \times V \longrightarrow \mathbb{Q}$ nondegen. alternating form, $(ax, y) = (x, \bar{a}y)$ $a \in \tilde{F}$
$G = GU(V, (\,,\,))$, fix $h : \mathbb{C}^* \longrightarrow G_{/\mathbb{R}}$ satisfying Riemann conditions
$\Rightarrow$ Shimura variety $Sh(G, h)_K$ $K \subset G(\mathbb{A}_f)$
defined over reflex field = field of definition of corresponding conjugacy class μ of cocharacters of G.

Fix prime p & make assumptions:
a. $K = K^p \cdot K_p$ K^p prime to p, $K_p \subset G(\mathbb{Q}_p)$ parahoric subgroup. ... e.g. $\Gamma_0(p)$
b. p remains prime P in F & is totally ramified [for convenience].
c. P unramified in $\tilde{F}/F$.

Fix embedding $\bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p$ & corresponding localization E_p of E
Want: integral model of $Sh(G, h)_K$ over $\mathcal{O}_{E_p}$, $Sh(G, h)_K^{\sim} \longrightarrow [\mathcal{M}/\mathcal{G}_{\mathcal{O}_E}]$ smooth, rel dim = dim G

$[\mathcal{M}/\mathcal{G}]$ = quotient stack of $\mathcal{M}$ = projective scheme: the local model for $(G_{\mathbb{Q}_p}, \{\mu\})$
$\mathcal{G}$ = smooth group scheme $/\mathbb{Z}_p$ with $\mathcal{G}(\mathbb{Z}_p) = K_p$

- Singularities of Sh & $\mathcal{M}$ are same for étale topology

New notation F_0 = discretely valued field with perfect residue field. F/F_0 totally ramified of degree e
π = uniformizer of F
V = F-vector space of dim. d, $\{e_1, \ldots e_d\}$ basis

$\Lambda_i = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}e_1, \dots \pi^{-1}e_i, e_{i+1}, \dots e_d\}$ $i = 0 \dots d$
part of periodic lattice chain

$I = \{i_0 < i_1 < \dots < i_{n-1}\} \subset \{0, \dots d-1\}$ nonempty

G = restriction $R_{F/F_0}(GL(V))$ $K_I = \mathrm{Stab}\{\Lambda_i : i \in I\}$
Also fix $(r_\varphi)_{\varphi \in \mathrm{Hom}_{F_0}(F, \bar{F}_0)}$ integers between $0, d$

$\longleftrightarrow$ conjugacy class μ of 1-parameter subgroups.

Local reflex field E: $\mathrm{Gal}(\bar{F}_0, E) = \{\sigma \in \mathrm{Gal}(\bar{F}_0/F) : r_{\sigma \circ \varphi} = r_\varphi \ \forall \varphi\}$

<u>Naive local model</u> $\mathcal{M}^{\text{naive}}/\mathcal{O}_E$: projective scheme, representing the functor

$S \longmapsto$ isom classes of commutative diagrams

$$\begin{array}{ccccccccc} \Lambda \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_S = \Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \dots & \to & \Lambda_{i_{n-1},S} & \xrightarrow{\pi} & \Lambda_{i_0,S} \\ \cup & & \cup & & & & & & \cup \\ \mathcal{F}_0 & \to & \mathcal{F}_1 & \to & \dots & & \to \mathcal{F}_{n-1} & \to & \mathcal{F}_0 \end{array}$$

$\mathcal{F}_i$ = locally direct summands as $\mathcal{O}_S$-modules, of rank $r = \sum_\varphi r_\varphi$, stable under $\mathcal{O}_F$ s.t.
characteristic polynomial of $i(a)$ on $\mathcal{F}_i = \prod_\varphi (T - \varphi(a))^{r_\varphi}$ $a \in \mathcal{O}_F$

- representable by projective scheme & $\mathcal{G}_{\mathcal{O}_E}$ acts :
$\mathcal{G} = \underline{\mathrm{Aut}}(\Lambda_i, i \in I)$

<u>Easy</u>: $\mathcal{M}^{\text{naive}} \otimes_{\mathcal{O}_E} E = \prod_\varphi \mathrm{Grass}_{r_\varphi, d - r_\varphi}$ product of Grassmannians as generic fiber

$$\mathcal{M}^{\text{naive}} \otimes_{\mathcal{O}_E} k \hookrightarrow \widetilde{\mathcal{F}}_I = LGL_d/\mathcal{P}_I \quad \text{affine flags}$$

- $\mathcal{P}_I$ stable subvariety.

(Görtz) <u>Theorem</u>! Let $e=1$ (unramified case). Then $\mathcal{M}^{\text{naive}}$ is flat $/\mathcal{O}_E$ with reduced special fiber. All irred components of special fiber are <u>normal</u> <u>Cohen-Macaulay</u> with rational singularities. At least for $d \leq 4$, $\mathcal{M}^{\text{naive}}$ is Cohen-Macaulay
Furthermore
$\mathcal{M}^{\text{naive}} \otimes_{\mathcal{O}_E} k = \bigcup_{w \in \mathrm{Adm}_I(\mu)} \mathcal{O}_w$ union of $\mathcal{P}_I$ orbits labeled by some $\mathrm{Adm}_I(\mu) \in \widetilde{W}_I \backslash \widetilde{W} / \widetilde{W}_I$

If $r=1 \Rightarrow \mathcal{M}^{naive}$ has semistable reduction (Drinfeld)

<u>Conjecture</u> $\exists$ $\mathcal{G}$-equivariant blowup of $\mathcal{M}^{naive}$ in the special fiber which has semistable reduction

- OK for $r=2$ (Faltings, Lafforgue)

<u>Theorem 2</u> Let $e \geq 2$. a. If r_φ's differ by at least $2 \Rightarrow \mathcal{M}^{naive}$ <u>not</u> flat$/\mathcal{O}_E$: dimension of generic & stable fiber not same!

b. If r_φ's differ by at most $1 \Rightarrow \mathcal{M}^{naive}$ is flat in the following cases: $r \leq e$, or $e=2$ or char $k=0$ (Weyman)

Let K = Galois closure of F/F_0, number embeddings $\varphi_1, \dots \varphi_e : F \to K$.

Splitting model $\tilde{\mathcal{M}}/\mathcal{O}_K$: projective scheme, represents functor:

$$
\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \to & \Lambda_{i_1,S} & \to & \cdots & \to & \Lambda_{i_{m-1},S} & \xrightarrow{\pi} & \Lambda_{i_m,S} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_0^e & \to & \mathcal{F}_1^e & \to & \cdots & \to & \mathcal{F}_{m-1}^e & \to & \mathcal{F}_0^e \\
\cup & & \cup & & & & & & \\
\mathcal{F}_0^{e-1} & \to & \mathcal{F}_1^{e-1} & \to & \cdots & \to & \mathcal{F}_{m-1}^{e-1} & \to & \mathcal{F}_0^{e-1}
\end{array}
$$

s.t. $\mathcal{F}$'s locally direct summand of rk $\mathcal{F}_i^s = \sum_{j=1}^{s} r_{\varphi_j}$

– analog of Demazure resolution in Grassmannian.

$(i(a) - \varphi_i(a))(\mathcal{F}_i^s) \subset \mathcal{F}_i^{s-1}$ $\forall i, s$

<u>Theorem 3</u> $\tilde{\mathcal{M}}$ is a twisted direct product of naive local models corresponding to GL_d/K, $\mu_i = \omega_{r_{\varphi_i}}$ minuscule coweight

$\tilde{\mathcal{M}} = \mathcal{M}^1 \tilde{\times} \mathcal{M}^2 \tilde{\times} \cdots \tilde{\times} \mathcal{M}^e$

(<u>Corollary</u> $\tilde{\mathcal{M}}$ is <u>flat</u> over $\mathcal{O}_K$.

Have $\tilde{\mathcal{M}} \to \mathcal{M}^{naive} \otimes_{\mathcal{O}_E} \mathcal{O}_K \to \mathcal{M}^{naive}$

– define the local model $\mathcal{M}$ as <u>image</u> of composed morphism

<u>Theorem 4</u> $\mathcal{M}$ is flat$/\mathcal{O}_E$ with reduced special fiber. Each irred component of special fiber is normal, CM with rational singularities.

Furthermore $\mathcal{M} \otimes_{\mathcal{O}_E} k = \bigcup_{w \in Adm(\mu)} \mathcal{O}_w$ $\mu = \mu_1 + \dots + \mu_e$

<u>Application</u>: vanishing cycles $R\Psi_K^{\mu} = R\Psi(\mathcal{M} \otimes_{\mathcal{O}_E} \mathcal{O}_K / \mathcal{O}_K)\, \mathbb{Q}_\ell\{d\}$

<u>Corollary</u> $R\Psi_K^{\mu} = R\Psi_K^{\mu_1} \times \ldots \times R\Psi_K^{\mu_e}$

<u>Remarks</u>: a. Same works for $G = R_{F/\mathbb{Q}}(GSp_{2n})$

b. $\exists$ problem for $G = GU$ corresp to $\tilde{F}/\mathbb{Q}$ where p is ramified in $\tilde{F}$.