G. Faltings Line bundles on the moduli of G-bundles

C proj smooth curve$/S$, geom. irred fibres
G simply connected simple semisimple (like E_8)
G-bundles E on $C \rightsquigarrow$ moduli stack $\mathcal{M}_G/S$

$\{\infty\}$ S-section, $\mathring{C} = C \setminus \infty$: any G-bundle P is locally (étale) on S trivial on $\mathring{C}$ & on $\hat{C}$
given by a section of LG (after choice of local eqn for ∞)

$LG(R) = G(R((t))) \supset L^{\geq 0}G(R) = G(R[[t]])$ groupscheme

$\cup$

$L^{\leq 0}G(R) = G(R[t^{-1}])$ ind group scheme

$\Rightarrow \mathcal{M}_G = \underbrace{\Gamma(\mathring{C}, G)}_{\text{ind scheme}} \backslash LG / L^{\geq 0}G$

$LG/L^{\geq 0}G = Y_G$ affine Grassmannian, $\operatorname{Pic} Y_G = \mathbb{Z}$,
generator:
$L^{-}G$-orbits on Y_G $\exists!$ open orbit through $*$
$\exists!$ codim 1 orbit H_θ θ = maximal root
Orbits classified by $W_G \backslash W_{aff} / W_G \simeq$ 1-parameter s.groups $Y_T^{\geq 0}$ in torus dominant

line bundle generator is $\mathcal{O}(H_\theta)$ (can define by filters by quotients which are fin dim scheme)
$\mathcal{O}(H_\theta)$ equivariant under central extension
$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{LG} \longrightarrow LG \longrightarrow 1$

Line bundles on $\mathcal{M}_G$ = $\Gamma(\mathring{C}, G)$-equivariant line bundles on Y_G ... $\Gamma(\mathring{C}, G)$ has no characters (Drinfeld)
$\Rightarrow$ line bundle can be equivariant in at most one way
$\Rightarrow \boxed{\operatorname{Pic} \mathcal{M}_G \subseteq \operatorname{Pic} Y_G = \mathbb{Z}}$

Main Theorem This is an isomorphism
$\longleftrightarrow$ have line bundle of "central charge" 1.

E G-representation, P G-bundle $\Rightarrow E_P$ vector bundle on C
$\Rightarrow \det(H^*(C, E_P)^{-1})$ gives line bundle on $\mathcal{M}_G$,

Central charge : trace form for E = $\pm$(central charge)$\cdot B$
B = minimal form : $B(H_\theta, H_\theta) = 2$

eg A_n or C_n get charge 1 from representations.

E orthogonal representation $\Rightarrow$ det $H^*(C, E_P)$ has a squareroot, the Pfaffian $\Rightarrow$ get charge 1 for all classical groups and G_2. $\rightarrow$ adjoint $\rightarrow$ Pfaffian for adjoint

eg E_8 get factor 60 or 30 : charge is that of smallest dim rep in general.

Assume G simply laced, $\alpha_1, \dots \alpha_\ell$ simple roots, $\varepsilon_1, \dots \varepsilon_\ell$ coroots, $a_{ij} = \alpha_i(H_{\alpha_j}) \in \{0, -1\}$

w_0 = maximal element in W_G.

Involution $\alpha_i^* = -w_0(\alpha_i)$ on roots

Given P on $C_{S'}$ for any basechange $S' \to S$ want to associate functorially a line bundle $\mathcal{L}_P$ on S' (of charge 1)

$\langle , \rangle : C_{S'} \to S'$ $\quad \mathcal{L}, \mathcal{M}$ line bundles on $C_{S'}$

$\Rightarrow \langle \mathcal{L}, \mathcal{M} \rangle$ line bundle on S' (Weil pairing)

$\quad \| $

$\det(H^*(C, ([\mathcal{L}]-[\mathcal{O}]) \otimes ([\mathcal{M}]-[\mathcal{O}]))$ in K-theory

[Other constructions of $\langle , \rangle$: 1. if $\mathcal{L} = \mathcal{O}(A)$ $\mathcal{M} = \mathcal{O}(B)$ & A, B coprime (not always possible if S not integral)

$\Rightarrow \langle \mathcal{L}, \mathcal{M} \rangle = \mathcal{O}(\pi_* A \cdot B)$ intersection product

2. $\mathcal{L} = \mathcal{O}(A)$, $\langle \mathcal{L}, \mathcal{M} \rangle = \mathrm{Norm}_{\pi/A}(\mathcal{M}/A)$

for $A \to S'$ finite : restrict to A & take norm.]

Assume P reduces to a B-bundle P' (B = Borel)

$\Rightarrow$ associated line bundles $\mathcal{L}_{\varepsilon_i}$ on $C_{S'}$ from characters $\varepsilon_i : B \to \mathbb{G}_m$

$$\mathcal{L}_{P'} = \bigotimes \langle \mathcal{L}_{\varepsilon_i}, \mathcal{L}_{\varepsilon_i} \rangle \otimes \bigotimes_{i<j} \langle \mathcal{L}_{\varepsilon_i}, \mathcal{L}_{\varepsilon_j} \rangle^{\otimes a_{ij}}$$

$\langle \mathcal{L}, \mathcal{M} \rangle \simeq \langle \mathcal{M}, \mathcal{L} \rangle$ naturally, on $\langle \mathcal{L}, \mathcal{L} \rangle$ map is $(-1)^{\deg \mathcal{L}}$

In additive notation write as $\sum_i \langle \mathcal{L}_{\varepsilon_i}, \mathcal{L}_{\varepsilon_i} \rangle + \sum_{i<j} \langle \mathcal{L}_{\varepsilon_i}, \mathcal{L}_{\varepsilon_j} \rangle a_{ij}$

Locally in étale topology on S $\exists$ reduction to Borel
- show $\mathcal{L}_{p'}$ independent of reduction
- rather show this for <u>generic</u> Borels:

$\alpha > 0 \Rightarrow \deg \mathcal{L}_\alpha \le -2g$ — generic Borels classified by a <u>smooth</u> $S' \twoheadrightarrow S$ (since $H^1(\mathcal{L}_\alpha^{-1}) = 0$)

$$\begin{tikzcd} X_p \arrow[r] & C_S \\ C_S \times S' \arrow[u, "\text{smooth}"] \arrow[ur] & \end{tikzcd}$$

$X_p \to C_S$ associated G/B bundle

S' is union of pieces of finite type over base

Have $\mathcal{L}_{p'}$ over S' : need to show has descent datum
$pr_1^* \mathcal{L}_{p'} \cong pr_2^* \mathcal{L}_{p'}$ on $S'' = S' \times_S S'$

Geometric Konstruction $X = G/B$; On $X \times X$ have $\mathcal{L}_{\mathcal{E}_i} \boxtimes \mathcal{L}_{\mathcal{E}_i}$ has unique (to scalar) G-invariant section s_i with divisor $\mathcal{D}_i$ = pairs of Borels in rel pos. $s_i w_0 = w_0 s_{i*}$

$\mathcal{L}_{\mathcal{E}_i - \alpha_i} \boxtimes \mathcal{L}_{\mathcal{E}_{i*}}$ has over $\mathcal{D}_i$ unique G-inv section t_i
$\Rightarrow$ sections on $X_{p_1} \times_C X_{p_2} \longrightarrow C_S$ of $\mathcal{L}_{\mathcal{E}_i} \boxtimes \mathcal{L}_{\mathcal{E}_{i*}}$,
& of restriction to $\mathcal{D}_i$.

Form $\langle s_i, t_i \rangle + \sum_{i<j} a_{ij} \langle s_i, s_j \rangle$

section of $\mathcal{L}^{(1)}_{\mathcal{E}_i - \alpha_i} \boxtimes \mathcal{L}^{(2)}_{\mathcal{E}_{i*}} |_{\mathcal{D}_i} \xrightarrow{\text{Norm}}$ section of $\langle \mathcal{L}^{(1)}_{\mathcal{E}_i - \alpha_i}, \mathcal{L}^{(2)}_{\mathcal{E}_{i*}} \rangle$ or something

Over S'' $\quad C_{S''} \xleftarrow{\text{smooth}} X_p \times_C X_p \longrightarrow C$

$\quad\quad\quad\quad\downarrow$
$\quad\quad\quad\quad S''$ $\quad \Rightarrow$ get section over S'' of $\sum \langle \mathcal{L}^{(1)}_{\mathcal{E}_i - \alpha_i}, \mathcal{L}^{(2)}_{\mathcal{E}_{i*}} \cdots$

$\rightsquigarrow$ get section of difference
$pr_1^* \mathcal{L}_{p_1} - pr_2^* \mathcal{L}_{p_2}$ which is isom (compute divisor) ϕ

Won't be descent data : s etc <u>only defined up to scalar</u> ...
On $S^{(4)} = S' \times_S S' \times_S S' \times_S S'$ $\quad \boxed{\phi_{13} \cdot \phi_{24} = \phi_{14} \cdot \phi_{23}}$

$pr_1^* \mathcal{L}_p \otimes pr_2^* \mathcal{L}_p \xrightarrow{\sim} pr_3^* \mathcal{L}_p \otimes pr_4^* \mathcal{L}_p$

$\Longrightarrow \phi_{11}^{-1} \circ \phi_{12}$ is a descent datum

Proof of $\phi_{13}\circ\phi_{24} = \phi_{14}\phi_{23}$: by degeneration argument

On $X \times X$ $\rho : \mathbb{G}_m \longrightarrow T$ dominant 1-param group
$\rho(t)(\mathcal{D}_i) \subset X \times X \times \mathbb{G}_m \subseteq X \times X \times \mathbb{P}^1$

$\hat{\mathcal{D}_i}$ closure in $X \times X \times \mathbb{P}^1$. Limit at $t = 0, \infty$:
diagonal $G \rightsquigarrow$ diagonal T + $N^+ \times N^-$

$\mathcal{D}_i$ becomes reducible, divisor $D_i \times X \cup X \times D_i^*$
t_i in limit vanishes on one component

Add 5^{th} Borel (after a faithfully flat base change :
assume P comes from B bundle & even T-bundle)
do degeneration over $C \times \mathbb{P}^1$, $X \times X \times X \times X \times \mathbb{P}^1$
as $t \to 0$, $t \to 1$ our ~~ratio~~ ratio of two compositions
is regular & goes to 1

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