R. Bezrukavnikov - Sheaves on affine flags & modular representations of Langlands dual Lie algebra

G simple algebraic group / $\mathbb{C}$

$\mathcal{G}r$ = affine Grassmannian $\quad \mathcal{F}\ell$ = affine flag variety

- indschemes / $\mathbb{C}$:

$\mathcal{G}r(\mathbb{C}) = G(\mathbb{C}((t))) / G(\mathbb{C}[[t]])$

$\mathcal{F}\ell(\mathbb{C}) = G(\mathbb{C}((t))) / I \quad$ Iwahori: $I \subset G(\mathbb{C}[[t]])$

$$\begin{array}{ccc} I & \subset & G(\mathbb{C}[[t]]) \\ \downarrow & & \downarrow \\ B & \subset & G \end{array}$$

Study categories of perverse sheaves on $\mathcal{G}r$, $\mathcal{F}\ell$. $\quad k = \bar{k}$ field of coefficients

Spherical sheaves:

$P_{sph} = Perv^k_{G(\mathbb{C}[[t]])}(\mathcal{G}r) \quad$ equivariant perverse sheaves

$P_I = Perv^k_I(\mathcal{F}\ell)$

<u>Geometric Satake Theorem</u> (Drinfeld, Ginzburg, Mirkovic-Vilonen)

P_{sph} is a tensor category under convolution & is equivalent to $Rep({}^LG)$ category of reps of Langlands dual group / k

(M-V.: general coefficient rings k)

We'll define a category P related to P_I for char $k = \ell \gg 0$, & a category R related to $Rep({}^L\mathfrak{g})$ Lie algebra reps s.t. $P \simeq R$. $\quad$ (P, R <u>abelian</u> categories)

Structure of argument: indirect - compute both sides ...

Argument will identify $D^b(P) \overset{\varphi_1}{\simeq} D^b(Coh^{{}^LG}(T^* {}^LG/{}^LB))$

$D^b(R) \underset{\varphi_2}{\simeq}$ $\quad$ equivariant coherent sheaves

The composition ϕ of φ_1, φ_2 will have better properties than φ_1, φ_2. $\quad \varphi_1$ variant of work jt. with S. Arkhipov

φ_2 joint with Mirkovic-Rumynin

(I) Definition of P. $\quad$ {isom classes of irred objects in P_I} $\leftrightarrow W$

$IC_w \leftarrow\!\shortmid w$

W = affine Weyl group

IC_w = intersection cohomology of I-orbit through w.

Let $W_f \subset W$ be finite Weyl group

$^f W$ = set of minimal reps of cosets in W/W_f =
$\{ w \in W \mid \ell(w_f \cdot w) \geq \ell(w) \ \forall w_f \in W_f \}$

$P :=$ Serre quotient of P_I with $^f W$ as set of irreducible objects: $P = P_I / \langle IC_w \mid w \notin {^f W} \rangle$

Remark 1. P is categorical (categorification) counterpart of antispherical module for the affine Hecke algebra $\mathcal{H}$

$K^0(P)$ has natural W action & as such is isomorphic to $\mathbb{Z}[W] \otimes_{\mathbb{Z}[W_f]} \text{sign}$ = antispherical module (induced rep of sign)

2. P has a Whittaker realization: $P \simeq$ category of Iwahori-Whittaker perverse sheaves (Bez.-Arkhipov)

Let $\mathcal{N} \subset {}^L\mathfrak{g}$ be the nilpotent cone, $\tilde{\mathcal{N}} = T^*({}^L G / {}^L B)$ Springer resolution of $\mathcal{N}$.

Theorem 1 (ℓ = char k = 0 or large) $\rightsquigarrow$ probably just > Coxeter # of ${}^L\mathfrak{g}$

$$D^b(P) \simeq D^b(\mathrm{Coh}^{{}^L G}(\tilde{\mathcal{N}}))$$

... isom on K-groups this induces $K^0(\mathrm{Coh}^{{}^L G}(\tilde{\mathcal{N}})) \simeq$ antispherical is well known - e.g. used in Kazhdan-Lusztig, see Chriss-Ginzburg.

(II) Definition of R: [char $k = \ell > 2h({}^L G) - 2 > 0$]

Start with category $\mathrm{Rep}\, {}^L\mathfrak{g} = \mathrm{Rep}\, U\, {}^L\mathfrak{g}$

Recall that the center Z of $U\, {}^L\mathfrak{g}$ contains two parts:

$Z \supset Z_{HC} = U({}^L\mathfrak{g})^{{}^L G} = \mathrm{Sym}(\mathfrak{h})^{W_f}$ $\quad \mathfrak{h}$ = Cartan

$\cup$ (Harish-Chandra center)

$Z_{Fr} = \langle x^\ell - x^{[\ell]} : x \in {}^L\mathfrak{g} \rangle \simeq \mathrm{Sym}\, \mathfrak{g}^{(1)}$ $\quad x^{[\ell]}$ = restricted power

(Frobenius center) $\rightsquigarrow$ Frob. twist

$$\mathrm{Spec}\, Z = \mathfrak{g}^{*(1)} \times_{(\mathfrak{h}^*/W)^{(1)}} \mathfrak{h}^*/W$$

where $\mathfrak{h}^*/W \longrightarrow \mathfrak{h}^*/W$ is Artin-Schreier

For $\lambda \in \mathfrak{h}^*$ let $\psi_\lambda : Z_{HC} \longrightarrow k$ be the corresponding character. For $\lambda \in \mathfrak{h}^*(\mathbb{F}_p)$, $Z \underset{Z_{HC}}{\otimes} \psi_\lambda \simeq \mathcal{O}(\mathcal{N})$

eg ψ_0 = character of trivial module
$\psi_{-\rho}$ = " " Steinberg module

$$R_0 = Rep_{f.g.}(U({}^L\mathfrak{g}) \underset{Z_{HC}}{\otimes} \psi_0) \qquad \text{finitely generated reps}$$

<u>Theorem</u> (B-Mirkovic-Rumynin) $\exists$ an Azumaya algebra $\mathcal{A}$ on $\tilde{\mathcal{N}}$ & an equivalence btw. $D^b(Coh_{\mathcal{A}}(\tilde{\mathcal{N}}))$ coherent sheaves of $\mathcal{A}$-modules $\overset{\wr}{}$ $D^b(R_0)$

<u>Rmk</u> $\mathcal{A}$ = (a version of) differential operators in characteristic p ("crystalline differops") — variant of Beilinson-Bernstein localization.

Let R_1 be the category of Harish-Chandra modules for the pair $({}^L\mathfrak{g} \oplus {}^L\mathfrak{g}, {}^LG)$ ⇢ diagonal where Z_{HC} acts by $\psi_0 \otimes \psi_{-\rho}$ regular char $\otimes$ most singular char.

<u>Remark</u> The action of $U({}^L\mathfrak{g} \oplus {}^L\mathfrak{g})$ on $M \in R_1$ factors through $\bar{U} = \left(U({}^L\mathfrak{g}) \underset{Z_{Fr}}{\otimes} U({}^L\mathfrak{g})\right) \underset{Z_{HC} \otimes Z_{HC}}{\otimes} (\psi_0 \otimes \psi_{-\rho})$

$\bar{Z}$ = Image of $Z \otimes Z$ in $\bar{U}$ can be described as $\bar{Z} \simeq \mathcal{O}(\mathcal{N})$, & for $\chi \in \mathcal{N}$ the reduction $\bar{U}_\chi = \bar{U} \underset{\bar{Z}}{\otimes} \chi$ is Morita equivalent to $U({}^L\mathfrak{g}) \underset{Z}{\otimes} (\psi_0, \chi)$

-- so for f.d reps doesn't matter if we work with U or $\bar{U}$.

<u>Theorem</u> $\underline{D^b(R_1) \simeq D^b(Coh^{{}^LG}(\tilde{\mathcal{N}}))}$

$D^b(R_1)$ carries a Serre-type duality & a self-dual t-structure

Duality is defined by $M \mapsto RHom_{\bar{U}}(M, \bar{U})$

"coherent perverse t-structure" : defined as follows

Def $M \in D^b(R_1)$ ies in $D^{p, \le 0}$ if for any ${}^L G$-orbit $X \subset N$, we have $M \otimes^L_{\mathcal{O}(N) = \bar{Z}} \mathcal{O}(X) \in D^{\le \frac{1}{2} \operatorname{codim} X}$

<u>Remark</u> Forgetful functor $F: R_1 \longrightarrow Coh^{{}^L G}(N^{(1)})$ $\hookrightarrow$ Fr-twist
(forget all but action of ${}^L G$ & $\bar{Z}$)
F sends $\mathbb{S}$ to Serre duality
& our t-structure to the t-structure of perverse coherent sheaves for middle perversity. (uniquely characterizes t-structure)
(cf math AG ...)

<u>Def</u> R = core of t-structure.

<u>Theorem</u> $D^b(R) \simeq D^b(P)$ sends P to R, $\mathbb{S}$ to Verdier duality
$\simeq D^b(Coh^{{}^L G}(\tilde{N}))$

<u>Remark</u> Closely motivated by results & conjectures of Lusztig!
(on characters of nonrestricted representations)

<u>Corollary</u> $\forall\, \chi \in N$ there is a subquotient (some quotient of abelian subcategory) P_χ of P which is equivalent to the category of H-(modules ($\bar{U}_\chi$, $Z(\chi)$ modules where $Z(\chi) \subset {}^L G$ is stabilizer of χ.

Idea of proof of Theorem 1 $D^b(P) \simeq D^b(Coh^{{}^L G}(\tilde{N}))$:

Basic ingredient 1) [Gaitsgory] <u>Central</u> <u>functor</u>
$Z: Rep\, {}^L G = P_{Sph} \longrightarrow P_I$.
V

2) [Gaitsgory] Z carries a tensor endomorphism : logarithm of monodromy ... $\Leftarrow$ Z defined via nearby cycles.

3) Wakimoto sheaf & filtration of $Z(V)$ by such (in char 0)

Z is a geometric counterpart of the isomorphism of Bernstein-Lusztig description of $Z(\mathcal{H})$ center of affine Hecke with $\mathbb{Z}[\Lambda]^{W_f}$ Λ = coweight lattice

Wakimoto sheaves: geometric counterparts of the embedding $\mathbb{Z}[\Lambda] \hookrightarrow \mathcal{H}$

$$\lambda \longmapsto q^{-\ell(\lambda)/2} T_\lambda \qquad \lambda \in \Lambda^+ \text{ dominant cone.}$$

$\mathbb{Z}[\Lambda]^{W_f} \subset \mathbb{Z}[\Lambda] \longleftrightarrow$ filtration by Wakimoto sheaves....

$Z(V)$ has a filtration by Wakimoto sheaves

$1, 2, 3 \xRightarrow[\text{(Tannakian arguments)}]{}$ a tensor functor $F : D^b(Coh^{G}\tilde{N}) \to D_I(F\ell)$

s.t. $V \otimes \mathcal{O} \longrightarrow Z(V)$

$\mathcal{O}(\lambda) \longmapsto$ Wakimoto sheaf

- check that F composed with projection functor to antispherical quotient is an equivalence.

Two parts - geometric Langlands & char p rep theory & composed equivalence is better for some reason

Replacing char $k = \ell$ by char 0 expect to replace modular reps by reps of quantum group at root of unity.