M. Emerton : p-adic automorphic representations

Motivating triangle:

$$\begin{array}{ccc} \text{Automorphic representations} & \longleftrightarrow & \text{Motives} \\ & \searrow\!\!\nwarrow \quad \swarrow\!\!\nearrow & \\ & \text{L-functions} & \end{array}$$

- $\pi = \bigotimes_v \pi_v$ automorphic rep. of $GL_n(\mathbb{A})$ (everything $/\mathbb{Q}$) s.t. π_∞ has algebraic infinitesimal character

$\big\downarrow$ local Langlands

$\{ (W_v' \longrightarrow GL_n) \}$ Weil(-Deligne) groups of finite places

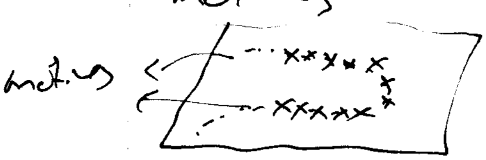
- n-dim motive $M \longrightarrow (W_v' \longrightarrow GL_n)$
  $\Uparrow$ Hodge theory for $v = \infty$, ℓ-adic theory $v = p$

L-functions from local L-factors, Γ-factors at ∞

: from motive: ℓ-adic cohomology for $\ell \neq p$ gives Weil-Deligne rep $W_p' \longrightarrow GL_n$ (Frobenius data, bad reduction)

For $\ell = p$ the decomposition group action $Gal(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$ on p-adic cohomology of M is more complicated - gives info on Frob, bad reduction & Hodge numbers - much more structure!

p-adic interpolation The reps $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{Q}_p)$ form a space (profinite group, pro p) parametrized by where we send profinite generators... won't all arise from motives - only crystalline ones can



Deformation space

- but motives don't sit in discrete fashion - Zariski closure conjecturally gives nice subvariety

$\Rightarrow$ find p-adic analytic families of reps interpolating motivic reps.

— Existence of nontrivial p-adic analytic families of reps $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_p)$ coming from motives.

e.g. n-dim motives can be twisted by finite order characters $\varepsilon : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}_p^*$

$$\begin{array}{ccc} \text{1-dim space of } \underline{\text{continuous}} \text{ characters} & \longleftarrow & \mathbb{Z}_p^* \times (\mathbb{Z}/N)^* \quad \text{(CFT)} \\ & & \text{(Iwasawa)} \end{array}$$

In fact have other directions of deformation.

Also special values $L(M,0)$ interpolate p-adically over these families (p-adic L-functions)
eg Kummer congruences for Dirichlet L-functions
- modulo conjectures can divide these through by a period to get algebraic numbers which interpolate.
eg Main conjecture of Iwasawa theory interpolates Beilinson conjecture

$\boxed{\text{Today!}}$ interpolate the <u>automorphic side</u> of the triangle

G = linear reductive connected alg group $/\mathbb{Q}$
$\pi = \pi_\infty \otimes \pi_f$ automorphic representation

<u>Def</u> π is <u>algebraic</u> if
- infl. character of π_∞ is algebraic, in fact equal to that of W, a f.dim rep of G
- π_f can be descended to some number field
(GL$_n$ at least: Clozel: 1st implies second)

(eg GL_2 k^{th} discrete series: character $\leftrightarrow$ that of k^{th} symmetric power)

Fix K, finite extension of $\mathbb{Q}_p$ s.t. W & π_f are defined$/K$

Define $\tilde{\pi} = (\pi_p \otimes W) \otimes \pi_f^p$ "classical p-adic automorphic rep attached to π"
rep of $G(\mathbb{A}_f)$ on K-vector space
(action on first factor is diagonal action of $G(\mathbb{Q}_p)$)

... analog of throwing away Loc, putting Hodge #
information into place of p.

<u>Context for such reps</u> (largely Schneider-Teitelbaum):

- $\tilde{\Pi}_p$ is <u>not</u> smooth if W is nontrivial: rather locally algebraic: any vector transforms under algebraic rep W under small enough $< G(\mathbb{Q}_p)$
- $$\tilde{\pi} = \varinjlim_{\substack{K^p \subseteq \mathbb{G}(\mathbb{A}_f^p) \\ \text{compact open}}} \tilde{\pi}^{K^p}$$

We're trying to p-adically interpolate: can't stay with locally algebraic:

Let V be a loc convex K-vectorspace with a continuous action of $G(\mathbb{A}_f)$.

<u>Def</u> V is an admissible locally analytic rep of $G(\mathbb{A}_f)$ if (I) for all compact open $K^p < G(\mathbb{A}_f^p)$ V^{K^p} is an admissible loc an rep of $G(\mathbb{Q}_p)$ in sense of Schneider-Teitelbaum.

- roughly finiteness condition (admissibility)
locally analytic: for given vector, on small enough open in G the orbit map to V is an analytic map. (Smooth case $\to$ usual sense of admissibility)

(II) $\varinjlim_{K^p} V^{K^p} = V$

Locally algebraic $\Rightarrow$ locally analytic.
Conversely if V is an admissible loc an rep of $G(\mathbb{A}_f)$, W is a fin dim algebraic rep of G, let V_W = closed subspace of V consisting of W locally algebraic vectors.

Look for admissible reps containing ∞ many π's (many nonzero V_W's) — being packed together continuously — the fundamental notion of p-adic interpolation of representations.

Idealised goal: To find an admissible loc analytic rep $\mathcal{A}$ of $G(\mathbb{A}_f)$ s.t. all $\tilde{\pi}$ (classical automorphic reps) embed into $\mathcal{A}$ and are identified with its subspace of locally algebraic vectors

Can achieve this if $G(\mathbb{R})$ is compact.

For other G get weaker versions of such
[$G(\mathbb{R})$ case: Shimura variety is just a finite set of pts $\Longrightarrow$ integral structure on automorphic forms
$H^0(\text{fin set}) = \text{fns}$ ... in general don't know this integral structure, rely on crutch of continuous cohomology]

The construction If $K_f \underset{\text{compact open}}{\subseteq} G(\mathbb{A}_f)$ let

$$Y(K_f) = \underset{\substack{\rotatebox{90}{$\subseteq$} \\ \text{fin union of symmetric spaces mod arithmetic grps}}}{G(\mathbb{Q})\backslash G(\mathbb{A})}/K_\infty^0 K_f$$

$K_\infty^0 \longrightarrow$ connected component of max cpt

fin union of symmetric spaces mod arithmetic grps
- eg GL_2 = fin union of modular curves

If W is an f.dim alg rep of $G/K \Rightarrow$ local system
$$\begin{array}{c} V_W = G(\mathbb{Q})\backslash\left((G(\mathbb{A}_f)/K_\infty^0 K_f) \times W^\vee\right) \\ \downarrow \\ Y(K_f) \end{array}$$
- form projective system in K

Apply interpolation paradigm: fix $n \geq 0$
$$H^n(V_W) := \varinjlim_{K_f} H^n(Y(K_f), V_W) \circlearrowleft G(\mathbb{A}_f) \text{ smooth rep}$$
(closely related to π_f's coming from automorphic forms)

If put subscript H^n_{cusp} breaks into all the algebraic π's ...

Def $$\hat{H}^n = K \underset{\mathcal{O}_K}{\otimes} \varinjlim_{K^p} \varprojlim_{r} \varinjlim_{K_p} H^n(Y(K_p K^p, \mathcal{O}_K/p^r))$$
locally convex inductive limit of sequence of Banach spaces
Carries continuous action of $G(\mathbb{A}_f)$

Properties: • passing to locally analytic vectors

$\tilde{H}^n_{l.a.}$ is an admissible locally analytic rep of $G(\mathbb{A}_f)$

proved using simplices calculation of cohomology / use triangles

• natural map $H^n(V_W) \otimes W \longrightarrow \tilde{H}^n_W \subseteq \tilde{H}^n_{l.a}$

• This map may not be isom but there is a spectral sequence

$$E_2^{ij} = H^i_W(\tilde{H}^j) \Rightarrow H^{i+j}(V_W) \otimes W$$

derived functors of $V \longmapsto V_W$ passing to locally algebraic vectors

— approximate forms of any weight in tame level by vectors of zero weight.

Also get Galois representations here using étale cohomology of Shimura varieties,

$G_{\mathbb{R}}$ compact: these are finite parts of automorphic reps

completely achieve goal.

Spectral theory Traditionally one interpolates Hecke eigenforms

- p-adic family living over characters of torus, Hecke eigenvalues move continuously

(ε in direction of local Langlands)

$\varepsilon \ll 1$: only consider principal series at p: finite slope

Assume G is quasisplit$/\mathbb{Q}_p$, choose $T \subset B \subset G/\mathbb{Q}_p$

$\hat{T}$ = rigid space of characters of $T(\mathbb{Q}_p)$

$$\hat{T}(E) = \mathrm{Hom}_{cont}(T(\mathbb{Q}_p), E^\times) = {}^L\mathrm{Hom}_{loc\,an}(W_p, {}^LT(E))$$

L-homomorphism   L-group

locally analytic maps of W_p to ${}^LT(E)$

[Fix K^p] — nontrivial on wild inertia

Eigensheaf of $\tilde{H}^n$ $\mathcal{E} \longrightarrow \hat{T}$ coherent sheaf with $\mathcal{H}(G(\mathbb{A}_f^p)/K^p)$ action

Main property: if $\psi \in \hat{T}$ s.t. $\psi = \underset{\text{integral dominant}}{\chi} \cdot \theta$ smooth then

letting W = h. weight rep with $hw = \chi$ & writing $\tilde{H}^n_W = \underset{\text{smooth}}{U} \otimes W$

we have $\mathcal{E}_\psi$ = (dual to) θ-eigenspace of Jacquet module of U (records maps to principal series)

[need slope$(\theta) + \rho \not\geq -s_\alpha(\chi + \rho)$ $\forall$ simple roots α