L. Lafforgue (III) Pavings of Polyhedra, gluing & compactification 2

$S \in S^{r,n} = \{(i_0, \dots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + \dots + i_n = r\}$ matroid polytopes

$E = E_0 \oplus E_1 \cdots \oplus E_n$ rk $E_\alpha \geq r - d^S_{\{0..n\}-\{\alpha\}} = r_\alpha$

$\overline{Gr}_S^{r,E} = \overline{\Omega}^{S,E}_\emptyset \subset \overline{\Omega}^{S,E}$ compactifications of thin Shubert cells

$\downarrow$

$A^S / A^S_\emptyset$

Functoriality • S' face of S $\Rightarrow$ natural morphism coming from taking induced paving on face

$$\begin{array}{ccc} \overline{\Omega}^{S,E} & \longrightarrow & \overline{\Omega}^{S',E} \\ \downarrow & & \downarrow \\ A^S / A^S_\emptyset & \longrightarrow & A^{S'} / A^{S'}_\emptyset \end{array}$$

• $\dim S = n - p$ codim p $\Rightarrow$ we have $\{0,..,n\} = J_0 \amalg \dots \amalg J_p$

$S = S^0 \times \dots \times S^p$

$$\begin{array}{ccc} \overline{\Omega}^{S,E} & \xrightarrow{\sim} & \overline{\Omega}^{S^0, E_{J_0}} \times \dots \times \overline{\Omega}^{S^p, E_{J_p}} \\ \downarrow & & \downarrow \\ A^S / A^S_\emptyset & \xrightarrow{\sim} & A^{S^0}/A^{S^0}_\emptyset \times \dots \times A^{S^p}/A^{S^p}_\emptyset \end{array}$$

[product decomposition of pavings]

• Duality isomorphism $E \longleftrightarrow E^*$

Theorem If $n+1 \leq 3$ or if $r = 2$ we have

1. $\overline{\Omega}^{S,E} \longrightarrow A^S / A^S_\emptyset$ is smooth
   $A^S / A^S_\emptyset$ is smooth if $n+1 = 2$ or if $r = 2$.
2. S' face of S $\overline{\Omega}^{S,E} \longrightarrow \overline{\Omega}^{S',E} \times_{A^{S'}/A^{S'}_\emptyset} A^S / A^S_\emptyset$ smooth (base change)

• e.g. $r=2$ rk $E_\alpha = 1$ $\forall \alpha$: $n+1$ points on $\mathbb{P}^1$, get compactification of $\overline{M}_{0,n+1}$ Grothendieck Knudsen & Kapranov - compactification of arbitrary # of points in general position in projective space, with strata labelled by pavings of polyhedra ..

• $r=2$ PGL_2^{n+1} / PGL_2 : assertion 1 is due to Faltings

$n=1$ PGL_r^n / PGL_r (DeConcini Procesi)

$r=2$ PGL_r^3 / PGL_r pavings of triangle (Lafforgue)

PGL_r^{nr} / PGL_r : Faltings introduces minimal models of projective space : $K = \mathrm{Frac}(A)$ A dvr, π uniformizer
$g_0 \dots g_n \in GL_r(K)$
$\longmapsto$ lattices $M_i = g_i(A^r) \subset K^r$, study their relative positions. $m_0, \dots m_n$ integers $\Rightarrow$ consider
$M = \pi^{m_0} M_0 + \dots + \pi^{m_n} M_n$
Take product of projective spaces $\prod \mathbb{P}(M)$ for all such - over K all identified
$\mathbb{P}(K^r)$, take closure:
= "Deligne scheme", has semistable reduction over special point.
Doesn't commute with base change for A.
$\Rightarrow$ modify special fiber, no longer semistable, but does commute with base change, consider projective system

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The toric variety of pavings with a distinguished face
S polytope, $\underline{S}$ paving $\Rightarrow$ cone $\mathcal{C}^S_{\underline{S}} \subset \mathbb{R}^S$
S' face of $\underline{S}$ $\Rightarrow$ cone $\tilde{\mathcal{C}}^S_{\underline{S}, S'}$ of functions
$\{ v : S \longrightarrow \mathbb{R}$ s.t. $v \in \mathcal{C}^S_{\underline{S}}$ & S' = set of points in S where v is minimized$\}$
Prop These cones $\tilde{\mathcal{C}}^S_{\underline{S}, S'} / \mathbb{R} \subset \mathbb{R}^S / \mathbb{R}$ are a fan
$\rightarrow \tilde{A}^S$ toric variety, torus $\tilde{A}^S_\emptyset = \mathbb{G}_m^S / \mathbb{G}_m$
$(\underline{S}, S') \longleftrightarrow$ orbits $\tilde{A}^S_{\underline{S}, S'}$.

Prop i) $\tilde{A}^S_\emptyset = \mathbb{G}_m^S / \mathbb{G}_m \twoheadrightarrow \mathbb{G}_m^S / (\mathbb{G}_m^S)_\emptyset = A^S_\emptyset$
$F: \tilde{A}^S \longrightarrow A^S$

ii) Projective, flat morphism ($\dim = \dim S$) and fibers are geometrically reduced.

$F : \tilde{A}^S \longrightarrow A^S$ Forgetful. $\underline{S}$ paving; $\alpha_{\underline{S}}$ distinguished point in corresponding torus orbit $\Rightarrow$
take fiber $Y_{\underline{S}}$ over $\alpha_{\underline{S}}$, carries action of $\mathbb{G}_m^{nr}$,

Lemma a. $Y_{\underline{S}} = \coprod \mathring{Y}_{S'}$ orbits labelled by S' = face of S

b. $\dim S' = n-p$ fixed by torus $(\mathbb{G}_m^{n+1})_{S'} \hookrightarrow \mathbb{G}_m^{J_0} \times \dots \times \mathbb{G}_m^{J_p}$

c. $Y_{S'} = \overline{\mathring{Y}_{S'}}$ proj. normal toric variety $\mathbb{G}_m^{n+1}/(\mathbb{G}_m^{n+1})_{S'}$

$\;\; = \coprod \mathring{Y}_{S'}$ $\quad S'$ faces of S.

Moduli problem $\quad \Omega^{S,E} \hookrightarrow A^S \times \mathbb{G}_m \backslash \prod_{i \in S} (\Lambda^r E_i - \{0\})$

Prop 7 $\quad \Omega^{S,E} \times_{A^S} \tilde{A}^S \longrightarrow Gr^{r,E}$ canonical morphism

respected by $\tilde{A}^S_\emptyset = \mathbb{G}_m^S/\mathbb{G}_m$

equivariant wrt action of $\mathrm{Aut}\, E_\bullet \supset (\mathbb{G}_m)^{n+1}$

$(F_{S'})_{S' \in \underline{S}} \in Gr_{\underline{S}}^{r,E}$ = fiber of $\Omega^{S,E}$ over $\alpha_{\underline{S}}$

$((F_{S'})_{S' \in \underline{S}}\, ;\, \alpha_{\underline{S},S'}) \longmapsto F_{S'}$

$\Rightarrow$ rank r equivariant v.b. $\mathcal{E}^S$ on $\Omega^{S,E} \times_{A^S} \tilde{A}^S$

$\longleftrightarrow$ $\mathbb{G}_m^{n+1}$ equivariant v.b. on quotient of this by $\tilde{A}^S_\emptyset$

which is $\overline{\Omega}^{S,E} \times_{A^S/\tilde{A}^S_\emptyset} \tilde{A}^S/\tilde{A}^S_\emptyset$

Def $\quad \overline{Vec}^{r,S}$ = algebraic stack over $A^S/A^S_\emptyset$, which associates to X scheme over $A^S/\tilde{A}^S_\emptyset$ the groupoid of vector bundles $\mathcal{E}$ of rank r, $\mathbb{G}_m^{n+1}$-equivariant on $X \times_{A^S/\tilde{A}^S_\emptyset} \tilde{A}^S/\tilde{A}^S_\emptyset$ verifying numerical conditions

Theorem $\quad \mathcal{E}^S$ defines a morphism

$$\begin{tikzcd}
\overline{\Omega}^{S,E} \arrow[r, "\text{smooth}"] \arrow[d] \arrow[dr, phantom, "\square"] & Vec^{r,S} \arrow[d] & \overset{\text{open}}{\subset} \overline{Vec}^{r,S} \\
\prod_{0 \le \alpha \le n} Gr^{r_\alpha, E_\alpha} \arrow[r] & \prod_\alpha BGL_{r_\alpha} &
\end{tikzcd}$$