M. Finkelberg - Uhlenbeck spaces for $\mathbb{A}^2$ & affine Lie algebra $\widehat{sl}_n$
(Drinfeld Gaitsgory Kuznetsov Braverman)

S surface : (Donaldson) Stable $SL_n$ bundles on $S =: Bun_G^a(S)$
$\longleftrightarrow$ ASD connections / gauge equivalence

($G = SL_n$, Chern class $c_2 = a \in H^4(S, \mathbb{Z}) = \mathbb{Z}$ fixed
= topological charge of connections)

Uhlenbeck : weak compactness - limits of connections
$\{A_i\} \longrightarrow A + \sum_{x_i \in S} m_i \delta_{x_i}$ $A$ = connection of topological charge $a - \sum m_i$, $\delta_{x_i}$ - $\delta$-like connections

$\Rightarrow \coprod_{b \leq a} Bun_G^b(S) \times Sym^{a-b} S$ (as topological space)

contains open part $Bun_G^a(S)$
What is algebrogeometric meaning of this space?
What is version for general simple simply-connected G?

We consider only $S = \mathbb{P}^2 \supset \mathbb{P}^1_\infty$ line at infinity, $\underline{G = SL_n}$.

$a = c_2(F)$ $Bun_G^a(\mathbb{A}^2)$ = fine moduli space of rank $n$ v. bundles on $\mathbb{P}^2$ with trivialization on $\mathbb{P}^1_\infty$, $F|_{\mathbb{P}^1_\infty} \xrightarrow{\sim} V \otimes \mathcal{O}_{\mathbb{P}^1_\infty}$
- smooth variety
($V$ $n$-dim vector space)

Linear algebraic description of $Bun_G(\mathbb{A}^2)$ : (ADHM
Barth, Ballinger...
Nakajima)

$V = \mathbb{C}^n$ $W = \mathbb{C}^a$ $V \underset{j}{\overset{i}{\rightleftarrows}} W$, $B_1, B_2 \circlearrowleft W$

Space of such data is
Symplectic, moment map $\mu$ to $gl(W)$
$\mu(B_1, B_2, i, j) = [B_1, B_2] + ij$

$\mu^{-1}(0) \supset \mu^{-1}(0)^s$ stable $(B_1, B_2, i, j)$ : Im $i$ generates $W$ under $B_1, B_2$
$\cup$ $\mu^{-1}(0)^c$ costable : if $\exists W' \subset \ker j$, $B_1 \& B_2$-invariant $\Rightarrow W' = 0$
stable, costable are open
$(\mu^{-1}(0)^s \cap \mu^{-1}(0)^c) / GL(W) = Bun_G^a(\mathbb{A}^2)$
free action

Construction $(B_1, B_2, i, j) \mapsto \mathcal{F} = H^{-1}(M)$ $M$ monad

$$M = \left( W \otimes \mathcal{O}(-1)_{\mathbb{P}^2} \xrightarrow{e} (W \oplus W \oplus V) \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{f} W \otimes \mathcal{O}(1)_{\mathbb{P}^2} \right)$$

$\mathcal{F} = H^{-1}(M)$ $\quad -2 \quad -1 \quad 0$

$e = (z_0 B_1 - z_1, z_0 B_2 - z_2, z_0 j)$ $\quad f = (-z_0 B_2 + z_2, z_0 B_1 - z_1, z_0 i)$

$z_0, z_1, z_2$ homog coords on $\mathbb{P}^2$, $\mathbb{P}^1_\infty = \{z_0 = 0\}$

Nakajima: Consider full $\mu^{-1}(0)$ with nonfree action

$\Rightarrow$ $\mu^{-1}(0) /\!/ G$ categorical quotient $(\operatorname{Spec} \mathbb{C}[\ ]^G)$

$Bun_G^a(\mathbb{A}^2) \subset N^a$ but $N^a$ has desired stratification:

$$N^a = \coprod_{0 \le b \le a} Bun_G^b(\mathbb{A}^2) \times Sym^{a-b}(\mathbb{A}^2)$$

- affine variety (due to framing)

What does $N^a$ classify?

Drinfeld-Bezrukavnikov: $(B_1, B_2, i, j)$ arbitrary $\mapsto M$ monad with several cohomologies

$H^{-1}(M) = \mathcal{F} \in D^b(Coh\, \mathbb{P}^2)$ is a perverse coherent sheaf

locally free in codimension 1

i.e. $i_x^* \mathcal{F} \in D^{\le 0}(Vect)$, $i_x^* \mathbb{D}\mathcal{F} \in D^{\le}(Vect)$

- singularities only at points, not curves...

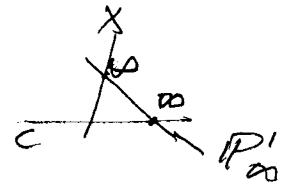
$\tilde{N}^a$ = stack $\mu^{-1}(0)/GL(W)$ = moduli of perverse coherent sheaves on $\mathbb{P}^2$, loc free in codim one, with triv at $\infty$, $c_2 = a$.

$\tilde{N}^a \longrightarrow N^a$ Nakajima variety is quotient of $\tilde{N}^a$ by equivalence relation $\tilde{N}^a \times_{N^a} \tilde{N}^a$: $\mathcal{F}_1 \sim \mathcal{F}_2$ iff $\exists$ third sheaf $\mathcal{F}_3 \xrightarrow{p_1} \mathcal{F}_1$, $\mathcal{F}_3 \xrightarrow{p_2} \mathcal{F}_2$ $\quad p_1, p_2$ generically isom,

cones of $p_1, p_2$ concentrated at finite subschemes of $\mathbb{A}^2$

$\det(\text{Cone } p_1) = \det(\text{Cone } p_2)$ in $Sym(\mathbb{A}^2)$

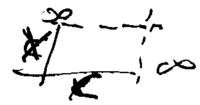
Another approach – for arbitrary groups:




blow up $\mathbb{P}^2$ at these two points

$$\mathbb{P}^2 \xleftarrow{p} Bl_{\infty,\infty} \xrightarrow{q} C \times X$$

the blow down



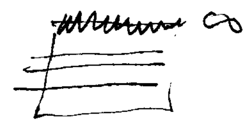

To $\mathcal{F}$ we associate $q_* p^* \mathcal{F}$ trivialized at $C \times \infty \cup \infty \times X$

Restriction to lines $c \times X$ we obtain a map $C \longrightarrow Bun_G(X)$

$\varphi(\infty)$ lies in open substack of trivial bundles on $X \simeq \mathbb{P}^1$ $\Rightarrow$ $\varphi(c)$ is trivial for almost all $c \in C$

Symmetrically $X \longrightarrow Bun_G(C)$ with same property

Similarly $\mathcal{F}|_{C \times x}$ trivial for almost all $x \in X$ $\Rightarrow$ only fin many lines where nontrivial $\rightarrow$ triv at $\infty$




extends to triv on a Zariski open nbhd of $C \times X$ – in particular to formal neighborhood

$$\Rightarrow \text{lift } \varphi : C \longrightarrow Bun_G(X)$$
$$C \longrightarrow \mathcal{G}_X \longrightarrow Bun_G(X)$$

$\mathcal{G}_X$: G-bundles on X with trivialization formally near $\infty \in X$

honest scheme $\longleftarrow \mathcal{G}_X$ (thick Grassmannian)

$$\Rightarrow Bun_G^a(\mathbb{A}^2) \xrightarrow{\sim} Maps^a((C,\infty),(\mathcal{G}_X, g))$$

$g$: the trivial bundle with trivial triv

$a \in H_2(\mathcal{G}_X) = \mathbb{Z}$ $\longleftrightarrow$ dual to $c_1(\mathcal{L}^a)$

det line bundle $\mathcal{L} = \mathcal{O}(\Delta)$, $\Delta$ = nontrivial bundles in $\mathcal{G}$.

$\Gamma(\mathcal{G}_X, \mathcal{L}) = V_{\omega_0}$ irreducible integrable basic representation

$\Rightarrow$ Plücker $\mathcal{G}_X \hookrightarrow \mathbb{P}(V_{\omega_0}^*)$

$Maps^a((C,\infty)(\mathcal{G}_X, g)) = \{ L \subset V_{\omega_0}^* \otimes \mathcal{O}_C$ line subbundles $: \deg L = -a,$ satisfy pointwise Plücker relations

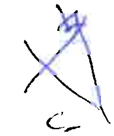
$\widehat{Maps}$ Drinfeld compactification: same condition at $\infty \in C$ $\}$

$= \{ L \subset V \otimes \mathcal{O}_C$ invertible subsheaves $+ \dots \}$

$\overline{Maps}^a = \coprod_{0 \le b \le a} Maps^b \times Sym^{a-b}(C \setminus \infty)$

- add configurations on the <u>line</u> rather than surface!

- very singular space & of infinite type

- replace as follows: $R$ = space of pairs of lines $(C_r, X_r)$ 

$Bun_G^a(\mathbb{A}^2) \subset \overline{Maps}(C, \infty, G_{X_r}, g)$ over $R$

$Uhl_G^a(\mathbb{A}^2) := \text{closure of } Bun_G^a(\mathbb{A}^2)$ : families of compatible quasimaps

$\parallel$

$\coprod_{b \le a} Bun_G^b(\mathbb{A}^2) \times Sym^{a-b}(\mathbb{A}^2)$ $\qquad (Sym\, \mathbb{A}^2 \longrightarrow Sym\, C)$

$IC(Uhl^a)$ : $\dim Bun_G^a(\mathbb{A}^2) = 2ah^\vee$

$\mathfrak{g} \rightsquigarrow \hat{\mathfrak{g}}^\vee$ Langlands dual affine algebra: dual Cartan matrix - exchanges some twisted & untwisted

e.g. $G_2^{(1)} \longrightarrow D_4^{(3)}$

Parabolic $\mathfrak{p} \subset \hat{\mathfrak{g}}^\vee$ $\quad \mathfrak{p} = \mathbb{C}\overset{\text{derivation}}{D} \oplus \mathfrak{g} \otimes \mathbb{C}[t] < \mathbb{C}D \oplus \mathfrak{g} \otimes \mathbb{C}(\!(t)\!)$

unipotent $\mathfrak{g} \otimes t\mathbb{C}[t] = \mathfrak{n}$ $\qquad \mathfrak{l} = \mathbb{C}D \oplus \mathfrak{g}$ Levi

$e \in \mathfrak{l}$ regular nilpotent $\Rightarrow$ monodromy filtration on $\mathfrak{n}$,

$\mathfrak{n}^e$ has filtration — view as graded space $\bigoplus_k (\mathfrak{n}^e)_k$

$\mathfrak{n}$ also graded by $\text{center}(\mathfrak{l}) = D$ i.e. by powers of $t$

$\Rightarrow \mathfrak{n}^e = \bigoplus_{k,d} (\mathfrak{n}^e)_k^d$

<u>Theorem</u> stalk of IC sheaf at a point $(\mathcal{F}, \sum m_i s_i)$

$= \bigotimes_i \left( \bigoplus_k Sym(\mathfrak{n}^e)_k^{m_i} [2k] \right) [2h^\vee b]$.

How does $\hat{\mathfrak{g}}^\vee$ arise? $\quad$ Replace $\mathcal{G}$ by flag variety $\mathcal{B} \to \mathcal{G}$

$\mathcal{B} \xhookrightarrow{\text{Plucker}} \mathbb{P}(\text{product of all fundamental reps})$

$H_2(\mathcal{B}, \mathbb{Z})$ = root ~~algebra~~ lattice of $\hat{\mathfrak{g}}^\vee$

So for $\alpha$ root of $\hat{\mathfrak{g}}^\vee$ can consider maps $(C, c) \to (\mathcal{B}, b)$ of degree $\alpha$

$\mathrm{Maps}^\alpha(C, c; \mathcal{B}, b) \quad \dim = 2|\alpha|$

$\downarrow \eta$

$(C \setminus \infty)^\alpha = \mathbb{A}^\alpha \qquad \eta$ = inverse image of Schubert divisor (colored) on $\mathcal{B}$

$\mathrm{Maps}^\alpha$ is symplectic, $\eta$ is an integrable system, & special fiber $\eta^{-1}(\alpha, 0) = \Lambda^\alpha$ Lagrangian subvariety in $\mathrm{Maps}^\alpha$. $\coprod_{\alpha} \mathrm{Irr}(\Lambda^\alpha)$ has crystal structure for $\mathfrak{g}^\vee$ (free crystal $B(\infty)$ of $U^+(\mathfrak{g}^\vee)$)

Drinfeld: Stack of perverse sheaves on a smooth proj surface (perverse sheaf which is v.b. outside finite set)

Flatness of families: wrt perverse t-structure.

Is this stack algebraic? probably yes .. should be because can consider arbitrary complexes: don't form stack since complexes have not just automorphisms but also self-exts $\mathrm{Ext}^i(\mathcal{I}, \mathcal{I})$ for $i < 0$ (at least fin many)

$\rightsquigarrow$ expect to get an $N$-stack, $N \sim \dim$ of variety (at least $\infty$ stack) should always be algebraic!

Grothendieck: coherents form honest alg stack

full derived should be $\infty$-stack & perverse things should be honest stack

Case of $\mathbb{P}^2$ with framing: ADHM/Nakajima descriptions as quotients — can consider as stack or as variety

$\rightsquigarrow$ coarse moduli very roughly