M.-F. Vigneras - Modular representations of p-adic groups

$\underline{G}$ = red. connected group / F local non arch. field

I Iwahori $\subset G = \underline{G}(F)$

(Borel $\left\{ \begin{array}{l} \text{Smooth } \mathbb{C} \text{ reps } V \\ \text{of } G \text{ generated by } V^I \end{array} \right\}$ is an abelian subcategory of Mod G, and the functor

$$V \longmapsto V^I = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}[I \backslash G], V)$$

[all reps today are smooth]

is an equivalence of categories between this and the category of right modules for the Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}}(G, I) = \mathrm{End}_{\mathbb{C}G}\, \mathbb{C}[I \backslash G]$.

Iwahori-Matsumoto : " affine Hecke algebra defined by W^{aff} with parameter q = order of residue field of F

Bernstein: this abelian subcategory is a direct factor in Mod G ... the unipotent block of G (for $\mathbb{C}$-reps) smooth reps

Kazhdan-Lusztig / Ginzburg: classify irreducible reps of $\mathcal{H}_{\mathbb{C}}(G, I)$ $\longleftrightarrow$ unipotent block

Q: How to replace $\mathbb{C}$ by a field R of positive characteristic? (or arbitrary field, algebraically closed)

Theorem $V \to V^I$ induces a bijection between irreducibles $V^I \neq 0$ (V = irr. smooth $\longleftrightarrow V^I$ = Simple $\mathcal{H}_R(G, I)$-module) if char $R \neq p$

Maybe true also in char $= p$: first iteration

Theorem True when char $R = p$ & $G = GL_2(\mathbb{Q}_p)$ & V has smooth central character

- can replace I by pro-p radical I_p :

$\{$ Smooth irr reps of $GL_2(\mathbb{Q}_p) \}$ $\longleftrightarrow$ $\{\underset{irr}{} \mathcal{H}_R(G, I_p)$ modules$\}$

Unipotent representations in unipotent block $\boxed{\text{char } R \neq p}$

Th. There is a unipotent block

i.e. trivial rep $\in$ indecomposable abelian category which is a direct factor :

$\mathrm{Mod}_R G = \mathrm{Block}_1 \times \mathcal{C}$, Block_1 indecomposable

Banal case: char R does not divide pro-order of any open compact subgroup — works just same as $\mathbb{C}$
"classical case" : char big enough also good

In general have difficulties
- Borel category (things gen by V^I) no longer abelian
- have unipotent reps V with $V^I = 0$.

... study using theory of types
First type: trivial representation of I
$s=$ type (P,τ): ~~other types~~ irreps τ of parahoric $P \supset I$ which are trivial on the pro-p radical P_1 & cuspidal on P/P_1 = finite reductive connected gp

We say s is contained in V if $V|_P \supset s$.
Theorem There exist an explicit finite set S of types such that the irred unipotents are exactly equal to the irreps containing a type $s \in S$

$s = (P,\tau)$ $\quad \bigoplus_s \mathrm{ind}_P^G \tau$: irred unipotents are quotients of such. In fact s is unique for given irreducible

(Valid also for $\mathbb{C}$-reps: have only one type, the trivial rep of I)
Q: what is endomorphism ring of these reps $\bigoplus_s \mathrm{ind}_P^G \tau$?

Unipotent block have to restrict to $GL(n,F)$
Let $\mathcal{H}_R(G)$ = global Hecke algebra : convolution algebra of $C_c^\infty(G,R)$

$$\mathrm{Mod}_R G \longleftrightarrow \mathrm{Mod}\, \mathcal{H}_R(G)$$

Two sided ideal $\mathcal{J} \subset \mathcal{H}_R(G)$: annihilator of $R[I \backslash G]$

Thm. V irreducible : the following are equivalent :

a. unipotent

b. subquotient of $R[I \backslash G]$

c. annihilated by J

and there exists an integer N ($\leq$ power of ℓ which divides $GL_n(\mathbb{F}_q)$) such that $(\mathrm{Mod}_R G)^{J^N} = \mathrm{Block}$,

(reps killed by J^N)

and $(\mathrm{Mod}_R G)^J \underset{\substack{\text{Morita}\\\text{equivalence}}}{\sim}$ modules for affine Schur algebra $\mathcal{S}$

$$\mathcal{S} = \mathrm{End}_{RG}\Big[\underbrace{\bigoplus_{I \subset P \text{ parahoric}} R[P \backslash G]}_{V}\Big] \qquad \text{induced reps for } P \text{ parahoric}$$

$$\mathcal{S} = \mathrm{End}_{RG} V \longrightarrow \mathrm{End}_{\mathcal{H}} V^I$$

Theorem $\mathrm{End}_{RG} V = \mathrm{End}_{\mathcal{H}_R} V^I$, double centralizer $(\mathcal{H}_R)$ = Schur!

$\mathcal{H}_R = \mathrm{End}_{\mathcal{S}_R} V^I$.

R.M. Green: constructed module W (over $\mathbb{C}$) for $\hat{U}_q \mathfrak{gl}_n$

quantum affine

$$\hat{U}_q(\mathfrak{gl}_n) \longrightarrow \mathrm{End}_{\mathbb{C}} W^n \quad \text{with image } \mathcal{S}_{\mathbb{C}}.$$

$$\mathrm{End}_{\mathcal{S}_{\mathbb{C}}} W^{\otimes n} = \mathcal{H}_{\mathbb{C}} \ , \quad V_{\mathbb{C}}^I \simeq W^{\otimes n} \text{ as } \mathcal{S}_{\mathbb{C}}\text{-modules}$$

(V as above from parahori)

Q: develop this theory over fields of char $p > 0$!

Proof: use Whittaker models – this is where GL_n is special /

Theory of types essential – reps induced from compact open subgroups.

1. The theorem of Borel & : irreds for Iwahori $\leftrightarrow$ G reps with I-fixed – obvious (to Schur?) $V \longrightarrow V^I$ bijection on irreds for $V^I \neq 0$ trivial, in fact for R with char R not dividing pro-order of I

Hard part: equivalence of categories !

2. Takeuchi: look at subgroup $I_p' = \left\{ \begin{pmatrix} 1 & 0 & & * \\ & \ddots & \ddots & \\ 0 & & & 1 \end{pmatrix} \right\} \subset$ prop radical I_p of I

use rep $V' = \mathrm{ind}_{I_p'}^{G} 1$ -- good for Whittaker model.

$\mathrm{End}_{\mathcal{H}_R}(V')^{I}$ is Morita equivalent to S_R

$\mathrm{End}_{RG} V' \twoheadrightarrow \mathrm{End}_{\mathcal{H}_R}(V')^{I}$: not isom but surjective

$V' = \mathcal{H}_R(G)\, e_{I_p'}$ e idempotent assoc to I_p'

So $\mathrm{End}_{RG} V' = \underbrace{e_{I_p'}\, \mathcal{H}_R(G) e_{I_p'}}_{J} \simeq S_R$

$\begin{pmatrix} (\mathrm{Mod}_R\, G)^{J=0} \\ \text{gen by } V^{I_p'} \neq 0 \end{pmatrix} \simeq \mathrm{Mod}\, S_R$

$V \longmapsto V^{I_p'}$

Whittaker model ⊢ fact shows all $(\mathrm{Mod}_R G)^{J=0}$ are gen by $V^{I_p'}$ ... $(V^{I_p'} \neq 0)$

...

[Prob. 6'> Irr V, $V^{I} \neq 0 \longleftrightarrow$ simple modules for $\mathcal{H}(G,P,\tau)$ & $\mathcal{H}(G,P,\tau) \sim$ generalized affine Hecke algebra]