D. Arinkin - Fourier transform for quantized integrable systems

1. Motivation : Geometric global quasiclassical Langlands
2. Quantized completely integrable systems

1. Geometric global Langlands in deRham setting :
   C smooth proj. curve $/\mathbb{C}$, G reductive group
   E local system for G on C $\rightsquigarrow$ Aut_E $\mathcal{D}$-module on $Bun\,{}^LG$
   Aut_E should be Hecke eigensheaf, eigenvalue E

Hope to glue all Aut_E as E varies to universal object
Aut = sheaf on $\mathcal{LS}_G \times Bun\,{}^LG$ $\qquad \mathcal{LS}$ = local systems = bundles + connection
really optimistic : hope Aut gives equivalence of derived categories $\mathcal{O}$-mod on $\mathcal{LS}$ $\quad$ $\mathcal{D}$-mod on $Bun\,{}^LG$.

Think of this as case $\lambda = 1$ & take range where λ close to 0.

<u>Def</u>. A λ-connection on a coherent sheaf $\mathcal{F}$ is a map
$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega \qquad \nabla(fs) = f\nabla s + \lambda s \otimes df$
$f \in \mathcal{O} \quad s \in \mathcal{F}$.
$\Rightarrow$ formulate λ-version of Langlands conjecture
$\mathcal{D}$-modules $\rightsquigarrow$ q-coh sheaves with λ-connection
$\longleftrightarrow$ $\mathcal{D}_\lambda$ λ-differential operators : $[\frac{\partial}{\partial x_i}, x_i] = \lambda$

<u>For $\lambda = 0$</u> $\qquad E = G$ λ-local system $= G$-Higgs bundle
Aut_E : module over $\mathcal{D}_0$ = sheaf of functions on $T^*Bun\,{}^LG$
$\rightsquigarrow$ coherent module on cotangent bundle.

Moduli of Higgs bundles $\simeq T^*Bun_G$

<u>Hitchin fibration</u>

$$\begin{tikzcd} T^*Bun\,G \arrow[dr] & & T^*Bun\,{}^LG \arrow[dl] \\ & Hitch & \end{tikzcd}$$

Generic fibers are dual abelian varieties (well really stacks, sometimes have so many components etc)
So one is Pic^0 of other so get a sheaf on each fiber - so Fourier-Mukai gives equivalence of derived category $\to$ throw away nonregular locus

Quantum completely integrable systems:

Def $\begin{matrix}(A,\omega)\\ \downarrow\\ B\end{matrix}$ classical completely integrable system = abelian scheme A over B with Lagrangian fibers
$\leftarrow$ Polarization of symplectic variety A.

Def A deformation quantization of (A,ω) is $\tilde{\mathcal{O}}_A$: sheaf of algebras (over $\mathbb{C}[[\lambda]]$) on A, with isom $i : \tilde{\mathcal{O}}_A / \lambda \tilde{\mathcal{O}}_A \xrightarrow{\sim} \mathcal{O}_A$, agreeing with symplectic structure: $i\left(\frac{f_1 f_2 - f_2 f_1}{\lambda}\right) = \{i(f_1), i(f_2)\}$

Ex $\mathcal{D}_\lambda$ for $\lambda \in \mathbb{C}[[\lambda]]$
as deformation of cotangent bundle

Def Quantized completely integrable system: $\begin{matrix}(A,\omega)\\ \pi\downarrow\\ B\end{matrix}$
$i : \tilde{\mathcal{O}}_A / \lambda \tilde{\mathcal{O}}_A \xrightarrow{\sim} \mathcal{O}_A$

$\pi^{-1}(\mathcal{O}_B) \longrightarrow \mathcal{O}_A$ Poisson commutative image

- would like to lift $\pi^{-1}(\mathcal{O}_B) \dashrightarrow \tilde{\mathcal{O}}_A \downarrow \mathcal{O}_A$: morphism of ringed spaces $\underset{\text{NC-scheme}}{(A, \tilde{\mathcal{O}}_A)} \longrightarrow (B, \mathcal{O}_B)$

Example: quantized Hitchin system: Beilinson-Drinfeld
TDO quantize T^*Bun_G & BD operators
lift $\mathcal{O}(\text{Hitch})$ to these TDO.

Duality for quantized completely integrable systems:
For $\left\{ \begin{matrix}(A,\omega)\\ \downarrow\\ B\end{matrix}, \tilde{\pi}^*: \pi^{-1}(\mathcal{O}_B) \longrightarrow \tilde{\mathcal{O}}_A \right\}$ quantized CIS $\Rightarrow$
consider "line bundles on fibers" of $\tilde{\mathcal{O}}_A$:
modules ℓ over $\tilde{\mathcal{O}}_A$, flat over $\mathbb{C}[[\lambda]]$, s.t. $\ell/\lambda\ell$ is a line bundle on a fiber $\pi : A \to B$

Ex. $b \in B$ with $m_b \subset \mathcal{O}_B$ maximal ideal, set
$\ell_b = \tilde{\mathcal{O}}_A / \tilde{\mathcal{O}}_A \cdot \tilde{\pi}^*(m_b)$ line bundle on fiber over b.

More generally given a $\mathbb{C}$-local system $\underset{\text{on fiber}}{E}$ [loc constant sheaf with 1-dim fiber]

⇒ twist our line bundle by $\mathcal{E}$ $\mathcal{L}_b \otimes_{\mathbb{C}} \mathcal{E}$

→ no obstructions to deforming a line bundle with Chern class 0:

use $\mathcal{O} \otimes \mathcal{E}$ the corresponding locally constant sheaf.

So for any $\begin{smallmatrix}\tilde{A}\\ \downarrow \\ \tilde{B}\end{smallmatrix}$ QCIS quant completely integrable system ⇒

dual $\tilde{A}^\vee$ = moduli space of quantized line bundles (commutative space) over $\mathbb{C}[[\lambda]]$, mod λ get usual line bundles on $\begin{smallmatrix}A\\ \downarrow \\ B\end{smallmatrix}$: special fiber is dual abelian scheme

·· doesn't sit over same base B

Langlands POV: dual should be moduli of local systems with small parameter, doesn't lie over Hitchin base.

Approximate description of $\tilde{A}^\vee$: Given $\begin{smallmatrix}A\\ \downarrow \pi \\ B\end{smallmatrix}$ A dim $2n$, B dim n

dim $3n$ $A^\natural$ = local systems on fibers = universal vector extension of $A^\vee$

$$A^\natural \to A^\vee \to B \qquad (\dim 3n,\ \dim 2n,\ \dim n)$$

Define foliations on $A^\natural$:

$\xi_1 = T(A^\natural / A^\vee)$,

ξ_2 = isomonodromic deformation of $A^\natural$ over B.

In fact $j: \xi_1 \simeq \xi_2$ as vector bundles (not as subbundles)

Claim For some quantization $\tilde{A}$, $\tilde{A}^\vee = A^\natural / (\xi_1 + \lambda \xi_2)$

$\xi_1 + \lambda\xi_2 = \{ (v, \lambda \cdot j(v)) \}$ using isom $j: \xi_1 \to \xi_2$

- quotient is scheme depending on λ. Quotient exists: reduce modulo λ & use flatness

Intuitively $\xi_1 + \lambda \xi_2$ describes isomonodromic deformation with "curvature" $= \frac{\omega}{\lambda}$

- replace bundles with connection by bundles with connection & given curvature.

Ex Constant CIS Vector space × Abelian variety ⟷ T^* abelian variety ⟷ geometric Fourier transform of Laumon, Polishchuk - Rothstein

Hope: quantized line bundles should be eigenline bundles for Hecke functors: make sense for Gl_n, seem to recover Langlands λ small in reverse direction

Similar description on Langlands dual side: description of moduli of λ-connections using same two-foliation picture, expect dual to be precisely moduli of λ-connections

Problem: there are many integrable system deformations! both B & $A^\vee$ ~~can~~ depend on λ