B Toën: Higher non-abelian Hodge theory

Purpose: Study the topological invariants of smooth projective complex variety

E.g.: H^*, π_1, π_i ...

Remark Non-abelian = unstable (deals with π_1, π_i ...)

Plan: 1) examples of the abelian HT

2) π_1 – theory

3) Higher degree (deg = 2)

1. Examples (Abelian case)

Thm (Hodge) X – smooth proj var $/\mathbb{C}$

$$\exists \quad H^n(X, \mathbb{C}) \underset{\text{natural iso}}{\cong} \bigoplus_{p+q=n} H^{p}(X, \Omega^q_X)$$

$\Omega^q_X = \Lambda^q \Omega^1_X$ $\quad$ Ω^1_X - holomorphic one form.

$$H^p(X, \Omega^q_X) = H^q(X, \Omega^p)$$

Cor.: $b_{2q+1}(X)$ are even.

$\Rightarrow$ restriction on the cohomology type of X.

Thm (Deligne) $X \xrightarrow{f} S$ smooth morph, X, S – smooth & projective. $/\mathbb{C}$.

$$H^n(X, \mathbb{C}) \longrightarrow H^n(X_s, \mathbb{C})^{\pi_1(S)}$$

$s \in S$ $\quad X_s = \pi^{-1}(s)$

It is a consequence of the degeneration of the Leray spectra sequence.

Restriction on morphisms of alg varieties

Main question of Non-abelian Hodge theory: What about unstable invariants.

e.g. $\pi_1(X)$

(Invariant is stable if it is on the stable homotopy category)

2. π_1 theory:

Remark: Hodge decomposition $H^n(X, \mathbb{C}) \simeq \oplus H^{pq}$ does not exist with rational coefficient (or for $\mathbb{Z}$ - coefficient)

$H^n(X, \mathbb{Z})$ is not algebraic enough
$\quad\quad\ \mathbb{Q}$ —

same for $\pi_1(X x)$

Still some parts of π_1 is algebraic
– the "linear part": $\pi_1(X, x) \xrightarrow{\rho} Gl_n(\mathbb{C})$
$\longrightarrow \{\mathrm{Imp}\}_{p,n}$ linear

Riemann-Hilbert correspondence:

$\{$lin. rep. of $\pi_1(X, x)\} \longleftrightarrow \left\{\begin{array}{l}\text{alg. vector bundles over } X \\ \text{w/ flat connections } \nabla: V \to V \otimes \Omega^1_X\end{array}\right.$

Remark (Toledo) $\exists$ alg $X/\mathbb{C}$ s.t. $\bigcap_{\rho : \pi_1(X) \to GL_n(\mathbb{C})} \ker \rho \neq \{e\}$

in this case lin part does not see the entire $\pi_1(X)$

Thm. (Non-abelian Hodge Correspondence) Corlette & Simpson

$$\{\text{lin rep's of } \pi_1(X)\} \longleftrightarrow \left\{ \begin{array}{l} \text{Higgs bundles } (V, \theta) \\ \text{s.t. semi stable} \\ c_1(V) = c_2(V) = 0 \\ \theta : V \to V \otimes \Omega^1 \\ \mathcal{O}\text{-linear}, \ \theta^2 = 0. \ (\text{flat}) \end{array} \right\}$$

(We think of these as degeneration of flat bundles)

is an equivalence of $\otimes$-categories.

$\mathbb{C}^*$ acts on $\{\text{Higgs bundles}\}$

$\lambda \in \mathbb{C}^* : (V, \theta) \mapsto (V, \lambda\theta)$.

By Thm $\mathbb{C}^*$ acts on the category of $\{\text{lin rep's.}\}$

$*$ $H^n(X, \mathbb{C}) \cong \bigoplus_{p+q} H^{p,q}$ can be seen as action of $\mathbb{C}^*$

of weight q on $H^{p,q}$

Def $\mathbb{C}^*$-action is called the Hodge decomposition of $\pi_1(X, x)$

ex. X = curve genus $= g$ sm. proj' rk $= 1$.

$$\left\{\begin{array}{c}\text{lin reps of rk} = 1\\ \text{of } \pi_1(X)\end{array}\right\} = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) =$$

$$\mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{C}^*) = (\mathbb{C}^*)^{2g}$$

$$\{L, \theta \mid \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X)\} \longleftrightarrow (\mathbb{C}^*)^{2g} \cong (S^1)^{2g} \times (\mathbb{R}^+)^{2g}$$

$$\| $$

$$(S^1)^{2g} \times \underset{\uparrow \text{ periods.!}}{\mathbb{C}^g} \circlearrowleft \mathbb{C}^*$$

$\mathbb{C}^*$ acts on $\{\text{rang } n \text{ lin rep's of } \pi_1(X,x)\} \longleftrightarrow \mathrm{Hom}(\pi_1(X), GL(\mathbb{C}))/GL_n$

$=$ moduli space of rk $= n$

Homotopy interpretation: $\exists$ pro-algebraic completion of the alg. group $\pi_1(X)$

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[dr] & \pi_1(X)^{alg} \arrow[d, "\exists !"] \\ & G \end{tikzcd}$$

$\pi_1(X)^{alg}$ = pro-alg. grp.

knowing $\{\text{lin reps of } \pi_1(X)\} \longleftrightarrow$ knowing $\pi_1(X)^{alg}$

Because $\mathbb{C}^*$ acting on $\pi_1(X)^{alg}$

Def The Hodge decomp. on π_1 is $\mathbb{C}^* \circlearrowleft \pi_1(X)^{alg}$

Remark $\pi_1^{alg}(X)$ = a homotopy invariant of X

$$\pi_1(X,x) \longrightarrow \pi_1(X,x)^{alg} \circlearrowleft \mathbb{C}^*$$

$$(=) \quad H^n(X,\mathbb{Z}) \hookrightarrow H^n(X,\mathbb{C}) \circlearrowleft \mathbb{C}^*$$

<u>Con.</u> (Simpson) $Sl_3(\mathbb{Z})$ is not π_1 of smooth proj variety/$\mathbb{C}$

3 Degree $= 2$

E.g. What about $\pi_2(X)$? as $\pi_1(X)$ module

Idea: consider a space T $\pi_1(T) = G$ alg. grp
$\pi_2(T) = V$ $\mathbb{C}$ vect. space
π_1 acts on algebraically on V $\pi_0 = *$

$\pi_0(Hom(X,T)) = [X,T]$

Remark $\pi_2(\cancel{X}) = 0 \iff [X,T] = Hom(\pi_1(X), G)/G$

$$X \xrightarrow{f} T \Rightarrow \left\{ \begin{array}{l} \pi_1(X) \to G \\ (\pi_2(X) \to V)_{\pi_1(X)\text{ equivar.}} \end{array} \right\}$$

<u>Thm</u> $\mathbb{C}^*$ acts on $\bigcup_T [X,T]$ = Hodge decomp. of deg $=2$ non-abelian coho.

$$[X,T] = \{ P \to X \quad G\text{-torsor} \quad \alpha \in H^2(X, P \times_G V)\}/\sim$$

There exist universal construction

$$\begin{tikzcd} X \arrow[r] \arrow[d] & (X \otimes \mathbb{C})^{sch}_{\leq 2} \arrow[dl, "\exists!\ alg."] \\ T & \end{tikzcd}$$

By universality, $\mathbb{C}^*$ acts on $(X \otimes \mathbb{C})^{sch}_{\leq 2}$

and $\pi_1(X\otimes\mathbb{C})^{sch}_2 = \pi_1^{alg}(X)$
$\pi_2(X\otimes\mathbb{C})^{sch}_2 = ?$ $\Big\}$ $\mathbb{C}^*$ action $\to$ Hodge decomposition

$$\begin{array}{c} \uparrow \\ \pi_2(X) \otimes \mathbb{C} \end{array}$$

Ultimately

$$X \longrightarrow (X \otimes \mathbb{C})^{sch} \circlearrowleft \mathbb{C}^{\times}$$

$\longrightarrow$ Hodge decomposition ~~of~~ on Schematization homotopy type of X.