Representations of surface groups in unitary groups with indefinite signature

Oscar García-Prada
CSIC, Madrid

March 2002, MSRI

Joint work with:

Steven Bradlow
Peter Gothen

Research note:

CRAS, Paris, August 2001

X closed oriented surface of genus $g \geq 2$

G algebraic group (real or complex)

Interested in:

reductive

$$\mathcal{R}(G) := \mathrm{Hom}^{+}(\pi_1(X), G) / G$$

algebraic variety

Basic questions

- Number of connected components of $\mathcal{R}(G)$?
- Non-emptiness ?

Answer is well-known for:

- G compact

  Narasimhan-Seshadri, Atiyah-Bott, Ramanan-Ramanathan, ...

- G complex reductive

  Hitchin, Simpson, Corlette, Donaldson...

- We study the case:

$$\boxed{G = PU(p,q)} \quad \left(= \frac{U(p,q)}{U(1)\,\mathrm{I}} \right)$$

- Previous work for non-compact real groups (non-complex)

- Goldman: $PSL(2,\mathbb{R})$
- Hitchin: $PSL(n,\mathbb{R})$, split forms
- Xia: $PGL(2,\mathbb{R})$, $U(p,1)$, $PU(2,1)$
- Gothen: $SU(2,2)$, $Sp(2,\mathbb{R})$
- Markman-Xia: $PU(p,p)$

Topological invariants for $PU(p,q)$ representations

$$\mathrm{Hom}(\pi_1(X), G)/G \leftrightarrow \left\{ \begin{matrix} (E,D) \text{ flat} \\ G\text{-bundles} \end{matrix} \right\} / \sim$$

Given $\rho \in \mathcal{R}(PU(p,q))$

$\mapsto$

flat $PU(p,q)$ - bundle

$\mapsto$ lift

(E, D)

$U(p,q)$-bundle

$U(p,q)$-connection with constant central curvature

Reduction to the maximal compact subgroup

$$U(p) \times U(q) \subset U(p,q)$$

determines:

$$E = V \oplus W$$

$$\mathrm{rk}\, V = p \;, \quad \mathrm{rk}\, W = q$$

Let

$$a = \deg V \;, \quad b = \deg W$$

the correspondence

$$\rho \longmapsto (a, b)$$

not well-defined since lift is <u>not</u> unique:

$$\begin{array}{l} E \longrightarrow \rho \\ E \otimes L \longrightarrow \rho \end{array}$$

any line bundle!

Invariant for ρ takes values in

$$\mathbb{Z} \times \mathbb{Z}/\mathbb{Z}$$

$$n.(a,b) = (a+pn,\, b+qn)$$

Have a map

$$R(PU(p,q)) \xrightarrow{c} \mathbb{Z} \times \mathbb{Z}/\mathbb{Z}$$

$$\rho \longmapsto c(\rho) = [a,b]$$

Consider the subspace

$$\mathcal{R}[a,b] = \left\{ \rho \;\middle|\; c(\rho) = [a,b] \right\}$$

Convenient to consider the central extension

$$0 \to \mathbb{Z} \to \Gamma \to \pi_1(X) \to 1$$

- One has:

$$\mathrm{Hom}(\Gamma, U(p,q)) / U(p,q) \leftrightarrow \left\{ \begin{array}{l} (E, D) \\ E : U(p,q)\text{-bundle} \\ D : \text{const. central curv.} \end{array} \right\} / \sim$$

- Consider:

$$\mathcal{R}_\Gamma(U(p,q)) := \mathrm{Hom}^+(\Gamma, U(p,q)) / U(p,q)$$

alg. variety.

- Topological invariants:

$$\mathcal{R}_\Gamma(U(p,q)) \xrightarrow{c} \mathbb{Z} \times \mathbb{Z}$$

$$E = V \oplus W \leftrightarrow \rho \longmapsto (a,b) = (\deg V, \deg W)$$

Can define the subspace

$$\mathcal{R}_\Gamma(a,b) = \left\{ \rho \in \mathcal{R}_\Gamma(U(p,q)) \,;\, \substack{\\ \\ c(\rho) = (a,b)} \right\}$$

Have

- $\mathcal{R}_\Gamma(a,b) \cong \mathcal{R}_\Gamma(a+pn, b+qn)$

- The map
$$\mathcal{R}_\Gamma(a,b) \longrightarrow \mathcal{R}[a,b]$$
is surjective.

$U(1)^{2g}$ - principal bundle.

Fact: $\mathcal{R}[a,b]$ is connected if and only if $\mathcal{R}_\Gamma(a,b)$ is connected.

Remark: if $b = -a$

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[dl, dotted] & 1 \\ U(p,q) & & \end{tikzcd}$$

Milnor – Wood type inequality

(Domic – Toledo)

Let $\rho \in \mathcal{R}[a,b] \subset \mathcal{R}(PU(p,q))$

and let

$$I = I(\rho) := 2\,\frac{qa - pb}{p+q}$$

Toledo invariant

well-defined for $[a,b]$

Fact:

$\mathcal{R}[a,b]$ is empty unless

$$\circledast \quad \boxed{|I| \le \min\{p,q\}\,(2g-2).}$$

(Same applies to $\mathcal{R}_\Gamma(a,b)$)

Some basic results

- $\mathcal{R}_\Gamma(a,b) \neq \phi$ for (a,b) satisfying the MW ineq.
- $\mathcal{R}_\Gamma(a,b)$ is <u>connected</u> if $GCD(p+q, a+b) = 1$

Let $\mathcal{R}^*_\Gamma \subset \mathcal{R}_\Gamma$ <u>irreducible reps.</u>

- For $|\tau| = \tau_{max}$

$$\mathcal{R}^*_\Gamma(a,b) = \phi \; ; \; \mathcal{R}_\Gamma(a,b) = \mathcal{R}^{red}_\Gamma(a,b)$$

- For any (a,b) so that $|\tau| \neq \tau_{max}$

$\mathcal{R}^*_\Gamma(a,b) \neq \phi$ and is <u>connected</u>

Ingredients of Proof:

Holomorphic point of view

- Higgs bundles
- Morse theory
- Holomorphic triples

– Give X the structure of a Riemann surface

– Higgs bundles (Hitchin, Simpson)

A Higgs bundle on X is a pair

(E, Φ)

- E holomorphic vector bundle
- $\Phi : E \to E \otimes K$ — canonical bundle of X

Higgs field

(E, Φ) is stable if

$\mu(E') < \mu(E)$

for all $0 \neq E' \subsetneq E \mid \Phi(E') \subset E' \otimes K$

$\left(\mu(E) = \deg E / \operatorname{rk} E \right)$

- semistable : replace "$<$" by "$\leq$"
- polystable : direct sum of stable ones of same slope

Theorem (Hitchin, Simpson, Corlette, Donaldson)

There is a homeomorphism

$$R_\Gamma(a,b) \cong \mathcal{M}(a,b)$$

which restricts to a homeomorphism

$$R_\Gamma^*(a,b) \cong \mathcal{M}^s(a,b)$$

Sketch of proof:

Fix a metric on X

ω : Kähler form

Fix $E = V \oplus W$ — C^∞

$U(p,q)$-bundle

reduction to a $U(p) \times U(q)$ (metrics on V and W)

$a = \deg V$

$b = \deg W$

$\mathcal{U}(p,q)$ - Higgs bundles

$$\begin{cases} E = V \oplus W \\ \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \qquad \begin{array}{l} \beta : W \to V \otimes K \\ \gamma : V \to W \otimes K \end{array} \\ \mathrm{rk}\, V = p \,,\ \mathrm{rk}\, W = q \end{cases}$$

<u>Let</u> $(a,b) \in \mathbb{Z} \times \mathbb{Z}$

$\mathcal{M}(a,b) :=$ moduli space of <u>polystables</u> $\mathcal{U}(p,q)$-Higgs bundles with $\deg V = a$, $\deg W = b$

$\mathcal{M}^s(a,b) \subset \mathcal{M}(a,b)$

stable

$$R_\Gamma(a,b) \cong \left\{ \begin{array}{l} D : U(p,q)\text{-connection on } E \\ F_D = D^2 = -i\mu I_E \omega \end{array} \right\} \Big/ \begin{array}{c} U(p,q)\text{-} \\ \text{gauge trans.} \end{array}$$

$\cong \Big\updownarrow$

Corlette-Donaldson $\Big\updownarrow \cong$ $D = \underbrace{d_A}_{U(p)\times U(q)} + \Theta$

$$\left\{ \begin{array}{l} (d_A, \Theta) \quad \text{s.t.} \\ F_A + [\Theta, \Theta] = -i\mu I_E \omega \\ d_A \Theta = 0 \\ d_A^* \Theta = 0 \quad \text{— harmonicity} \end{array} \right\} \Big/ \begin{array}{c} U(p)\times U(q) \\ \text{gauge trans.} \end{array}$$

$\Big\updownarrow \cong$ $\Theta = \Phi + \Phi^*$

$$\mathcal{M}(a,b) \underset{\text{Simpson}}{\overset{\text{Hitchin}}{\cong}} \left\{ \begin{array}{l} (d_A, \Phi) \,.\ \text{s.t.} \\ F_A + [\Phi, \Phi^*] = -i\mu I_E \omega \\ \bar{\partial}_A \Phi = 0 \end{array} \right\} \Big/ \begin{array}{c} U(p)\times U(q) \\ \text{gauge trans.} \end{array}$$

Have to study $\mathcal{M}(a,b)$.

- $\mathcal{M}^s(a,b) \subset \mathcal{M}(a,b)$

  smooth complex variety of dim

  $$1+(p+q)^2(g-1)$$

- MW inequality is a consequence of semistability:

  Necessary for $\mathcal{M}(a,b)$ non-empty

for $\tau = 2\,\dfrac{qa-pb}{p+q}$

$$|\tau| \leq \min\{p,q\}(2g-2) = \tau_{max}$$

- From semistability we obtain also information for $|\tau| = \tau_{max}$

$|\tau| = \tau_{max}$ & $p \neq q$ $\begin{pmatrix} \text{assume} \\ p > q \end{pmatrix}$

$$\mathcal{M}^s(a,b) = \phi$$

Every element in $\mathcal{M}(a,b)$ is reducible.

In fact:

$$\mathcal{M}(p,q,a,b) := \mathcal{M}(a,b) \cong$$

$$\mathcal{M}(q,q,a,a-q(2g-2)) \times M(p-q, b-a+q(2g-2))$$

This is a Higgs bundle version of a result on:

$\mathcal{R}(PU(p,1))$ – Toledo

$\mathcal{R}(PU(p,2))$ – L. Hernández

for $|\tau| = \tau_{max}$.

- dimension of $\mathcal{M}(p,q,a,b)$ is smaller than expected! τ_{max}
- Non-emptiness & conn. $\Leftrightarrow$ $\mathcal{M}(q,q,a,a-q(2g-2))$

Morse theory on $\mathcal{M}(a,b)$ (Hitchin)

S^1- action on $\mathcal{M}(a,b)$

$$(E,\Phi) \longmapsto (E, e^{i\theta}\Phi)$$

Moment map:

$$f : \mathcal{M}(a,b) \longrightarrow \mathbb{R}$$

$$f(E,\Phi) = \int_X |\Phi|^2 \, d\mathrm{vol.}$$

- f is proper (Uhlenbeck)

$\Rightarrow$
- f has a minimum on each connected component of $\mathcal{M}(a,b)$.
- If the subspace of local minima is connected then $\mathcal{M}(a,b)$ is connected

Local minima

Theorem The local minima of $f : \mathcal{M}(a,b) \to \mathbb{R}$ is given by

$$\mathcal{W}(a,b) := \left\{ (E,\Phi) \mid \beta = 0 \text{ or } \gamma = 0 \right\}$$

(Recall $E = V \oplus W$, $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$)

More precisely:

- $\gamma = 0$ if $\frac{a}{p} \leq \frac{b}{q}$ $(\tau \leq 0)$
- $\beta = 0$ if $\frac{a}{p} \geq \frac{b}{q}$ $(\tau \geq 0)$
- $\beta = \gamma = 0$ iff $\frac{a}{p} = \frac{b}{q}$ $(\tau = 0)$

– Holomorphic triples (Bradlow, G–P)

$T = (E_1, E_2, \varphi)$, where

$E_1, E_2 \to X$ holomorphic vector bundles

$\varphi : E_2 \to E_1$

α-slope: $\alpha \in \mathbb{R}$

$$\mu_\alpha(T) := \mu(E_1 \oplus E_2) + \alpha \frac{\mathrm{rk}\, E_2}{\mathrm{rk}(E_1 \oplus E_2)}$$

Def. $T = (E_1, E_2, \varphi)$ is α-stable if

$$\mu_\alpha(T') < \mu_\alpha(T)$$

for all proper subtriples $T' = (E_1', E_2', \varphi')$

$$\begin{tikzcd} E_2 \arrow[r, "\varphi"] & E_1 \\ E_2' \arrow[u, hook] \arrow[r, "\varphi'"] & E_1' \arrow[u, hook] \end{tikzcd}$$

– usual notions of semistability & polystability

There is a <u>moduli space</u>

$$\mathcal{N}_\alpha = \mathcal{N}_\alpha(n_1, n_2, d_1, d_2)$$

of α-polystable triples $T=(E_1, E_2, \varphi)$

with $\begin{cases} n_1 = \mathrm{rk}\, E_1, \; n_2 = \mathrm{rk}\, E_2 \\ d_1 = \deg E_1, \; d_2 = \deg E_2 \end{cases}$

- complex analytic variety
- projective when $\alpha \in \mathbb{Q}$

Theorem (Bradlow, G.P)

$$\mathcal{N}_\alpha \longleftrightarrow \left\{ \begin{array}{l} \text{Solutions to} \\ i\Lambda F_{A_1} + \varphi\varphi^* = \tau_1 I_1 \\ i\Lambda F_{A_2} - \varphi^*\varphi = \tau_2 I_2 \end{array} \right\} \Big/ \text{gauge}$$

vortices

$$\begin{cases} \alpha = \tau_1 - \tau_2 \\ n_1\tau_1 + n_2\tau_2 = d_1 + d_2 \end{cases}$$

- Dimensional reduction of Hermitian Yang-Mills on $X \times \mathbb{P}^1$

Range for non-emptiness of $\mathcal{W}_\alpha(n_1, n_2, d_1, d_2)$

(Can assume $n_2 \leq n_1$)

Let $\mu_1 = \frac{d_1}{n_1}$, $\mu_2 = \frac{d_2}{n_2}$

Define:

$$\alpha_m := \mu_1 - \mu_2$$

$$\alpha_M := \frac{2n_2}{n_1 - n_2}(\mu_1 - \mu_2), \quad n_1 \neq n_2$$

Proposition $\mathcal{W}_\alpha(n_1, n_2, d_1, d_2)$ is empty unless:

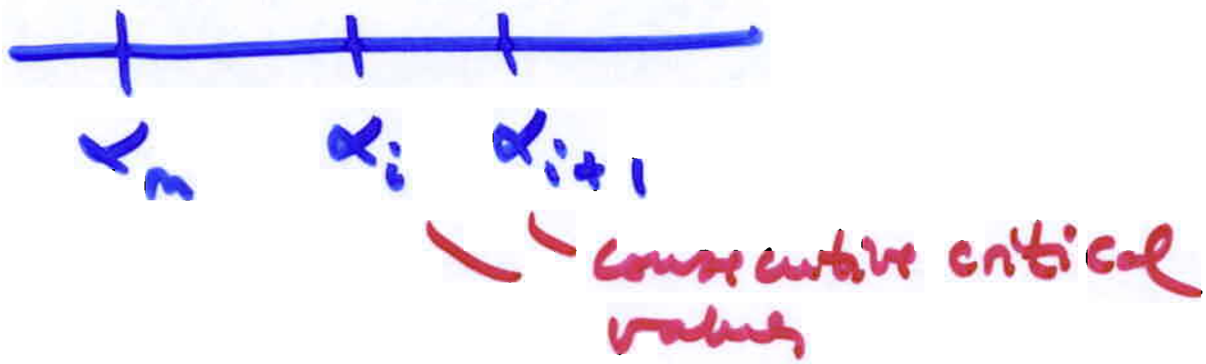
- $0 \leq \alpha_m \leq \alpha < \infty$ if $n_1 = n_2$
- $0 \leq \alpha_m \leq \alpha \leq \alpha_M$ if $n_1 \neq n_2$

Critical values

α's for which there is $T' \subset T \in W_\alpha$

s.t. $\mu_\alpha(T') = \mu_\alpha(T)$

- discrete set $\subset [\alpha_m, \infty)$ if $n_1 = n_2$
- finite set $\subset [\alpha_m, \alpha_M]$ if $n_1 \neq n_2$



$W_\alpha \simeq W_{\alpha'}$ if $\alpha, \alpha' \in (\alpha_i, \alpha_{i+1})$

If
- α is non-critical (generic)
- $GCD(n_2, n_1+n_2, d_1+d_2) = 1$

Then

$$\underbrace{W_\alpha^s}_{\text{stable}} = W_\alpha$$

Local minima and triples

Suppose $\frac{a}{p} \leq \frac{b}{q}$ $(\tau \leq 0)$ ~ similar for $\tau \geq 0$

$$\mathcal{W}(a,b) = \left\{ E = V \oplus W, \ \Phi = \begin{pmatrix} 0 & \beta \\ \underbrace{0}_{\gamma = 0} & 0 \end{pmatrix} \right\}$$

$$\overset{1-1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{triples} \\ W \xrightarrow{\beta} V \otimes K \\ \text{of type } (n_1, n_2, d_1, d_2): \end{array} \right\}$$

$$\left[\begin{array}{l} n_1 = p, \ n_2 = q \\ d_1 = a + p(2g-2), \ d_2 = b \end{array} \right.$$

Proposition

$\left(V \oplus W, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right)$ (poly)stable (as a Higgs bundle)

$\Longleftrightarrow$ the triple $(V \otimes K, W, \beta)$ α-(poly)stable for $\alpha = 2g-2$.

- $$\boxed{\mathcal{W}(a,b) \cong \mathcal{W}_{2g-2}(p, q, a + p(2g-2), b)}$$

α-Range and Toledo invariant

Let (n_1, n_2, d_1, d_2) be type of triples in $\mathcal{N}(a,b)$ — minima

- $2g-2 \geqslant \alpha_m$

  $2g-2 = \alpha_m \iff \tau = 0$

$p=q$ • $\alpha_m \geqslant 0 \iff |\tau| \leqslant p(2g-2)$

$p\neq q$ • $2g-2 \leq \alpha_M \iff |\tau| \leqslant \min\{p,q\}(2g-2)$

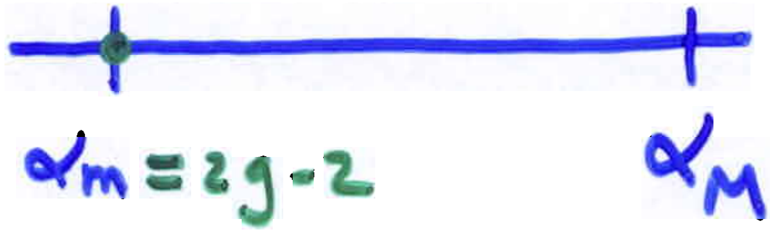
$2g-2 = \alpha_M \iff |\tau| = \min\{p,q\}(2g-2)$

Hence:

Necessary conditions for existence of triples and Higgs bundles are the same.

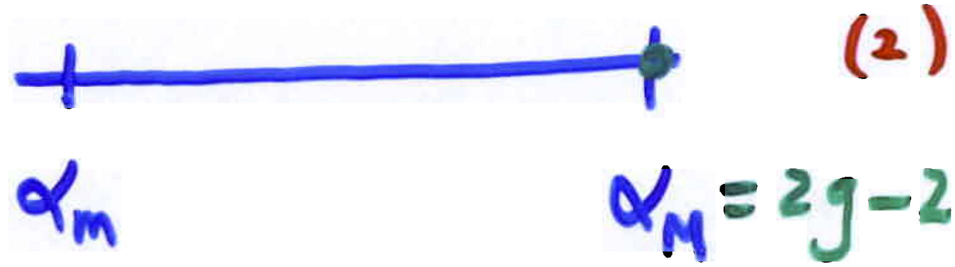
Case $p \neq q$

- $\tau = 0$



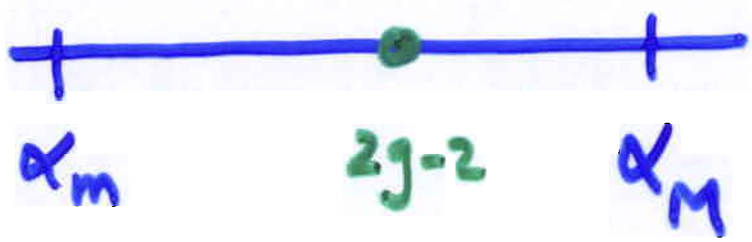
(1)

- $|\tau| = \tau_{max}$



(2)

- $0 < |\tau| < \tau_{max}$



(3)

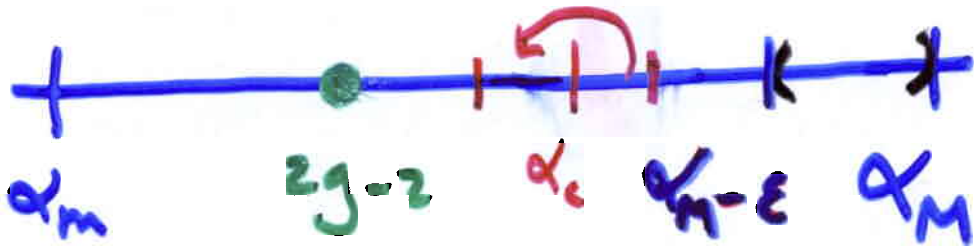
(1) : $\quad \mathcal{N}_{\alpha_m} \simeq M(n_1, d_1) \times M(n_2, d_2)$

(2) : $\quad \mathcal{N}_{\alpha_M} \overset{\text{birational}}{\dashrightarrow} M(n_2, d_2) \times M(n_1 - n_2, d_1 - d_2)$

$n_2 < n_1$

- smaller dim. than expected
- irreducible.

Large α moduli space $(n_2 < n_1)$



Thaddeus

$$\mathcal{N}_L := \mathcal{N}_\alpha(n_1, n_2, d_1, d_2), \quad \alpha \in (\alpha_M - \varepsilon, \alpha_M)$$

Theorem

$(E_1, E_2, \varphi) \in \mathcal{N}_L^s \Rightarrow$

- $0 \to E_2 \xrightarrow{\varphi} E_1 \to F \to 0$, F locally free
- E_2, F semistable bundles

Conversely: E_2, F stable $\Rightarrow$ (E_1, E_2, φ) α-stable for $\alpha \in (\alpha_M - \varepsilon, \alpha_M)$.

Conclusion:

$\mathcal{N}_L^s$ is birationaly equivalent to a $\mathbb{P}^N$-fibration over

$$M(n_2, d_2) \times M(n_1 - n_2, d_1 - d_2)$$

In particular:

- $\mathcal{N}_L^s$ non-empty
- irreducible

Theorem

$\mathcal{N}_\alpha^s$ is birationaly equivalent to $\mathcal{N}_L^s$

for $\alpha \geq 2g-2$.

- Codimension counting at critical α's
- Obstructions ($\mathbb{H}^2$) vanish for $\alpha \geq 2g-2$ (true for general triples!)
- $\mathcal{N}_\alpha^s$ smooth for $\alpha \geq 2g-2$

Hence

$\mathcal{N}^{s}_{2g-2}$ irreducible,

$\Rightarrow$ $\mathcal{M}^{s}(a,b)$ connected

If $(p+q, a+b) = 1$

$\Rightarrow$ $\mathcal{M}^{s}(a,b) = \mathcal{M}(a,b)$

- $\mathcal{M}(a,b)$ connected

$\Leftrightarrow$ $R_{\Gamma}(a,b)$ connected.

$\Leftrightarrow$ $\mathcal{R}[a,b]$ connected.