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Branched hyperbolic structures and rank 2 Higgs bundles

S - closed oriented surface, genus $g > 1$

$\pi = \pi_1(S)$

$\pi_0(\mathrm{Diff}^+(S)) \cong \mathrm{Out}(\pi)$ acts on $\mathrm{Hom}(\pi, G)/G$

say for $G = \mathbb{P}SL(2, \mathbb{R})$

If we choose a Riemann surface M diffeomorphic to $S \Rightarrow$ get a map

$$\mathrm{Hom}(\pi, G)/G \longrightarrow \begin{pmatrix} \text{Higgs bundles of} \\ \text{rank 2 on } M \end{pmatrix}$$

Question: How does this map vary when we vary M in the Teichmuller space $\mathcal{T}_S$ of S.

Generally $\mathrm{Hom}(\pi, G)/G$ has an invariant symplectic structure (independent of the choice of conformal structure on S).

The symplectic structure gives rise to a $Out(\pi)$-invariant measure on $Hom(\pi, G)/G$ and we get a unitary representation of $Out(\pi)$ on $L^2(Hom(\pi, G)/G)$.

__Fact__: When G is compact it follows that the action of $Out(\pi)$ on $Hom(\pi, G)/G$ is ergodic (i.e. $L^2(Hom(\pi, G)/G)$ has no invariant vectors). In fact one can show that $L^2(\text{---})$ has no invariant finite dimensional subspaces. (weak mixing)

__Question__: Is $L^2(Hom(\pi, G)/G)$ an irreducible representation of $Out(\pi)$?

__Example__: $S = T^2$, $G = U(1)$

$$Out(\pi) = SL(2, \mathbb{Z})$$

$$Hom(\pi, G) = \mathbb{R}^2/\mathbb{Z}^2$$

The action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2/\mathbb{Z}^2$ is by multiplication and if we take

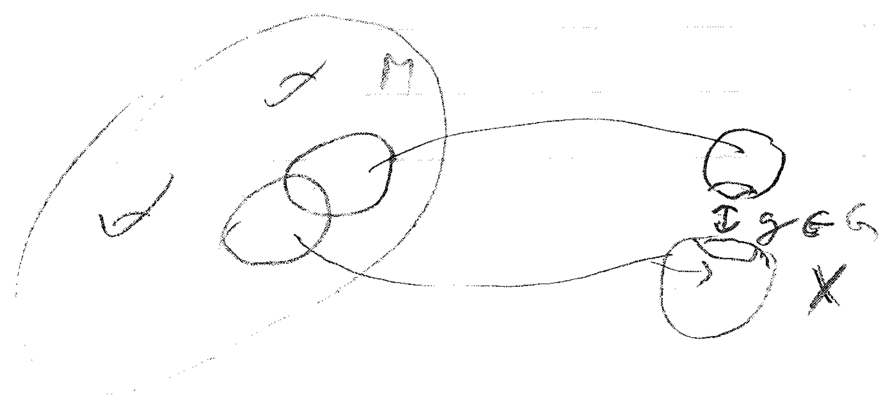
$$mult_d : L^2(\mathbb{R}^2/\mathbb{Z}^2) \to L^2(\mathbb{R}^2/\mathbb{Z}^2)$$

$\Rightarrow$ the range is an invariant subspace.

Example: G – compact (e.g. $G = U(n)$)
$\rightsquigarrow$ get the Narasimhan – Seshadri description of principal $G_{\mathbb{C}}$ bundles

Other extreme: Uniformization of S – geometric structures.

General yoga: Fix a local model $X = G/H$ – symmetric space.
Consider a space M and locally defined maps from opens of M into X + require that the transition functions are in G.



$\rightsquigarrow$ get a map
$$\text{dev} : \tilde{M} \to X$$
called the developing map and a representation $\pi_1(M) \to G$ making dev equivariant.

We can now look at

$$\mathrm{Def}_{(G,X)}(S) \to \mathrm{Hom}(\pi, G)/G$$

local homeo onto an open set.

<u>For instance:</u>

If $X = H^1_{\mathbb{C}}$, $G = PU(1,1)$

$$\begin{array}{c} \mathcal{T}_S \hookrightarrow \mathrm{Hom}(\pi, G)/G \\ \text{\rotatebox{90}{$\cong$}} \\ \mathbb{R}^{6g-6} = \mathbb{C}^{3g-3} \end{array}$$

<u>Fricke:</u> $\mathrm{Out}(\pi)$ acts properly discontinuously on $\mathcal{T}_S$ and $\mathcal{T}_S / \mathrm{Out}(\pi)$ is the ~~m~~ moduli of conformal structures on S.

$$\mathcal{T}_S \leftrightarrow \{[\varphi] \mid \varphi \text{ - embedding onto a discrete subgroup of } G = PU(1,1)\}$$

G - non-compact semisimple group
$G \supset PU(1,1)$ or $SU(1,1)$

$\Rightarrow$ Out(π) does not act ergodically on $\mathrm{Hom}(\pi_1, G)/G$

Example: If $G = \mathbb{P}GL(3,\mathbb{R}) \cong SL(3,\mathbb{R})$
$\Rightarrow$ $\mathrm{Hom}(\pi_1 G)/G$ has 2 components (Hitchin).

One component $\longleftrightarrow$ convex $\mathbb{RP}^2$ structures on S.

Hitchin: Identifies with $H^0(M, K^2) \oplus H^0(M, K^3)$
In general $\mathcal{C}(S)$ fibers over $\mathcal{T}_S$ with fiber $H^0(M, K^3)$

John Loftin: (affine spheres) $\mathcal{C}(S) \cong \mathbb{R}^{16g-16}$ with proper action of Out(π).

Hitchin: For any split real form G $\Rightarrow$ $\mathrm{Hom}(\pi_1 G)/G$ has a component diffeo to $\mathbb{R}^N$.

Conjecture: Out(π) acts properly on the Hitchin - Teichmüller component

Example: $G = SL(2, \mathbb{C}) \Rightarrow$

$QF = \left\{ \begin{array}{l} \text{embeddings onto quasi-} \\ \text{Fuchsian groups} \end{array} \right\} \xrightarrow{\text{Bers}} \mathcal{T}_S \times \overline{\mathcal{T}_S}$

(Open neighborhood of the Fuchsian representations) has a proper action of $\mathrm{Out}(\pi)$.

What can we say about other components of $\mathrm{Hom}(\pi, \underbrace{\mathbb{P}SL(2, \mathbb{R})}_{\mathbb{P}U(1,1)}) / \mathbb{P}U(1,1)$

$G = \mathbb{P}U(1,1)$

$$\begin{array}{ccl} \mathrm{Hom}(\pi, G)/G & \xrightarrow{e} & \mathbb{Z} \\ \varphi & \longrightarrow & \underset{\uparrow \text{ Euler class}}{e}\left(\begin{array}{l} \text{oriented } H^1_{\mathbb{C}} \\ \text{bundle on } S \end{array} \right) \end{array}$$

Milnor-Wood inequality

$$|e(\varphi)| \leq |\chi(S)| = 2g-2$$

Moreover

$$e(\varphi) = \pm \chi(S)$$

$\Leftrightarrow$ φ - discrete embedding e.g. for $g=2$ we get 5 - components

-2 -1 0 1 2

singular

$\mathcal{T}_{\mathcal{S}}$ $\overline{\mathcal{T}}_{S}$

<u>Facts:</u> • $e^{-1}(j)$ - connected $\forall j$

• $e^{-1}(2-2g+k) \longrightarrow \mathbb{C}^{3g-3-k}$ bundle over $\mathrm{Sym}^k(M)$

(Hitchin) for all $0 \le k < 2g-2$.

<u>Conjecture:</u> $\mathrm{Out}(\pi)$ acts ergodically on $e^{-1}(2-2g+k)$ $\forall k>0$.

<u>Remark:</u> Hitchin's $\mathbb{C}^{3g-3-k}$ bundle over $\mathrm{Sym}^k(M)$ has a zero section which parameterizes the so called branched hyperbolic structures on $\mathcal{S}$

Let $\mathrm{Sym}^k(\mathcal{S}) \to \mathcal{T}_{\mathcal{S}}$ be the relative k-th symmetric power of the universal curve over $\mathcal{T}_{\mathcal{S}}$.

Hitchin's inclusion

$$\mathrm{Sym}^k(M) \overset{\text{zero section}}{\hookrightarrow} \mathbb{C}^{3g-3-k} \text{ bundle over } \mathrm{Sym}^k(M) = e^{-1}(2-2g+k)$$

globalizes to give

$$\begin{array}{ccc} \mathrm{Sym}^k(\mathcal{S}) & \xrightarrow{\ \sigma\ } & e^{-1}(2-2g+k) \\ \downarrow & & \\ \mathcal{T}_{\mathcal{S}} & & \end{array}$$

which is a homotopy equivalence

<u>Note</u>: a k-branched hyperbolic structure is a singular Riemannian metric on the surface M with area element

$$-dA + \sum_{i=1}^{k} (2\pi - \theta_i)$$

cone angle at the i-th point

Relationship with Higgs bundles:

Consider a C^2 bundle E on M with a flat connection ∇.

Look for a Hermitian metric H on E an write

$$\nabla = \nabla_H + \psi$$

the H unitary connection

self adjoint w.r.t. H

$\nabla_H^{(0,1)} = \nabla''$ is a new holomorphic structure on M

Write $\psi = \underset{(1,0)}{\Phi} + \underset{(0,1)}{\Phi^*}$ parts for ∇''.

Self-duality equations read:

$$\begin{cases} \nabla'' \Phi = 0 \\ F(\nabla_H) + [\Phi, \Phi^*] = 0 \end{cases}$$

Since (E, ∇) - flat $\Rightarrow$

$$E = \tilde{M} \times_h \mathbb{C}^2$$

$$h : \pi_1(M) \to GL(2, \mathbb{C})$$ - the monodromy repr.

In these terms H can be viewed as a map

$$f : \tilde{M} \to Her(\mathbb{C}^2)$$
$$\| $$
$$SL(2, \mathbb{C}) / SU(2) = H^3_{\mathbb{R}}$$

$\Phi = D' f$ and self-duality $\Leftrightarrow$ f - harmonic as a map to $H^3_{\mathbb{R}}$.

<u>Uniformization:</u> Representing M as the quotient

$$M = H^1_{\mathbb{C}} / \pi_1(M)$$

We can construct a stable Higgs field over M as follows.

Choose $K^{\frac{1}{2}}_M$ - theta characteristic and let $E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$

A point $x \in H^1_{\mathbb{C}}$ can be thought of as a negative line in the Hermitian vector space $\mathbb{C}^{1,1}$. This gives a map

$$H^1_{\mathbb{C}} \xrightarrow{\lambda} \text{lines in } \mathbb{C}^{1,1}$$

or alternatively a $U(1,1)$-invariant line bundle inside $H^1_{\mathbb{C}} \times \mathbb{C}^{1,1}$ (Hermitian metric on E).

Thus we get $K \otimes \mathrm{End}(E) \supset K \otimes \underbrace{\mathrm{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}})}_{= K^{-1}}$

$\Rightarrow \Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ gives rise to a stable Higgs field.

More generally if $D \in \mathrm{Sym}^k(M)$ is an effective divisor one can modify

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & \psi \\ s & 0 \end{pmatrix}$ where s - section vanishing on D
ψ - quadratic differential vanishing on D.

<u>Note</u>:

$$\mathrm{Sym}^k(\mathcal{S}) \xrightarrow{\sigma} e^{-1}(2-2g+k)$$

is $\mathrm{Out}(\pi)$ equivariant and:

- $\mathrm{Out}(\pi)$ acts properly discontinuously on $\mathrm{Sym}^k(\mathcal{S})$
- $\mathrm{Out}(\pi)$ acts ergodically on $e^{-1}(2-2g+k)$ (conjecturally).

<u>Thm</u>: (Goldman-Neumann) If $k=1$ $\Rightarrow$ σ is surjective

<u>Remark</u>: If $k \geq 2$ $\Rightarrow$ σ is not surjective