Higgs bundles on Riemann Surfaces

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Representations of Fundamental Group

Genus g: The number of holes.

$$\pi_1(X) = \{A_i, B_i : 0 \leq i \leq g, \\ \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = e\}$$

G: a linear group, e.g. $GL(n, \mathbb{C})$, $U(n)$, $U(p,1)$ and etc.

$Hom(\pi_1(X), G)$: set of homomorphisms from $\pi_1(X)$ to G.

$$Hom(\pi_1(X), G) = \{(a_1, b_1, ..., a_g, b_g) \in G^{2g} : \\ \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = I\}$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow 0.$$

$$Hom(\Gamma, G) = \{(a_1, b_1, ..., a_g, b_g) \in G^{2g} : \\ \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \in Z(G)\}$$

Representations (continue)

G acts on $Hom(\Gamma, G)$ by conjugation (isomorphism) and we form the quotients $Hom^{+}(\Gamma, G)/G$ (of completely reducible representations)

Suppose G is $U(p,q)$ with $p \leq q$. Then each representation is associated with two invariants:

Chern class: $c \in \mathbb{Z}$ and $0 \leq c < p+q$;

Toledo invariant: $\tau \in \frac{2}{p+q}\mathbb{Z}$ and

$-2q(g-1) \leq \tau \leq 2q(g-1)$.

Theorem 1 *(Xia)*
$Hom^{+}(\pi_1(X), U(p,1))/U(p,1)$ *has one connected component for each even integer* $\tau \in [-2(g-1), 2(g-1)]$.

Theorem 2 *(Markman,Xia)*
$Hom^{+}(\Gamma, U(p,p))/U(p,p)$ *has one connected component for a fixed* c *and for each Toledo invariant* $2(p-1)(g-2) < \tau \leq 2p(g-1)$.

$GL(n,\mathbb{C})$-Higgs bundles

$GL(n,\mathbb{C})$-Higgs bundles: (V,Φ) where V is a vector bundle and $\Phi : V \to V \otimes \Omega^1$. Let (V,Φ) be a Higgs bundle.

Slope: $\mu(V) = deg(V)/rank(V)$

(Semi-)Stability: If $W \subset V$ and $\Phi(W) \subset W \otimes \Omega^1_X$, then $\mu(W)(\leq) < \mu(V)$.

Denote by $\mathbb{C}M$ the moduli of S-equivalence of semi-stable $GL(n,\mathbb{C})$-Higgs bundles.

Higgs bundles and the representation varieties

Theorem 3 (Hitchin et al) *$Hom^{+}(\Gamma, G)/G$ and $\mathbb{C}M$ are homeomorphic.*

Example ($G = \mathbb{C}^*$):

$$\mathbb{C}^{\cdot} : \mathbb{C} \to 0 \to \cdots; \quad \Omega^{\cdot} : \Omega^0 \to \Omega^1 \to \cdots$$

$$0 \longrightarrow H^1(\Omega^0) \longrightarrow H^1(\mathbb{C}) \longrightarrow H^0(\Omega^1) \longrightarrow 0,$$

$$\Omega^{0\cdot} : \Omega^0 \to 0 \to \cdots; \quad \Omega^{0,\cdot} : \Omega^{0,0} \stackrel{\overline{\partial}}{\to} \Omega^{0,1} \to \cdots$$

$$H^1(\Omega^0) = H^{0,1}_{\overline{\partial}}.$$

$$0 \longrightarrow \mathbb{Z}^{\cdot} \longrightarrow \mathbb{C}^{\cdot} \longrightarrow \mathbb{C}^{*\cdot} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z}^{\cdot} \longrightarrow \Omega^{\cdot} \longrightarrow \Omega^{*\cdot} \longrightarrow 0.$$

$$0 \longrightarrow H^1(\mathbb{Z}^{\cdot}) \longrightarrow H^1(\mathbb{C}^{\cdot}) \longrightarrow H^1(\mathbb{C}^{*\cdot}) \longrightarrow H^2(\mathbb{Z}^{\cdot}) \cdots$$

$$0 \longrightarrow H^1(\mathbb{Z}^{\cdot}) \longrightarrow H^1(\Omega^{\cdot}) \longrightarrow H^1(\Omega^{*\cdot}) \longrightarrow H^2(\mathbb{Z}^{\cdot}) \cdots$$

The $\mathbb{C}^*$-action

The $\mathbb{C}^*$-action:

$$\mathbb{C}^* \times \mathbb{C}M \to \mathbb{C}M, \quad (t, (V, \Phi)) \mapsto (V, t\Phi).$$

The fixed points of this action are called Hodge bundles.

Chern class: $c = deg(V)$. Denote by $\mathbb{C}M_c \subset \mathbb{C}M$ the subspace associated with c.

Proposition 1 $\mathbb{C}M_c \cong \mathbb{C}M_{-c}$.

Isomorphism : $(V, \Phi) \to (V^*, \Phi^*)$.

$U(p,q)$-Higgs bundles

$U(p,q)$-Higgs bundles: $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ where V_P, V_Q are vector bundles of rank-p and rank-q and

$$\Phi_1 : V_P \to V_Q \otimes \Omega^1_X, \quad \Phi_2 : V_Q \to V_P \otimes \Omega^1_X$$

(Note: $\mathfrak{u}(p,q) = \mathfrak{u}(p) \times \mathfrak{u}(q) + \mathfrak{p}$).

Denote by $M \subset \mathbb{C}M$ the subspace of $U(p,q)$-Higgs bundles. The C^*-action preserves M.

The Toledo invariant

Let $(V_P \oplus V_Q, \Phi) \in M$.

Toledo invariant: $\tau = \frac{qd_P - pd_Q}{p+q}$.

Denote by $M_{d_P,d_Q} \subset M$.

WOLOG $\tau \geq 0$ or equivalently $d_P \geq d_Q$.

The C^*-action preserves M_{d_P,d_Q}.

Binary Hodge bundles

A $U(p,q)$-Higgs bundle $(V_P \oplus V_Q, (\Phi_1, \Phi_2))$ is binary if $\Phi_2 \equiv 0$. Denote by $B_{d_P,d_Q} \subset M_{d_P,d_Q}$ the subscheme of binary Hodge bundles.

Binary Hodge bundle: $(V_P \xrightarrow{\Phi} V_Q \otimes \Omega^1_X)$

Proposition 2 *Every Higgs bundle in M can be deformed to a binary Hodge bundle.*

Hodge bundles and Moorse theory

Theorem 4 (Hitchin et al) *The Hodge bundles are critical points of $\mathbb{C}M$ (M_{d_P,d_Q}).*

Compute the Moorse index at critical points.

$$End(V) \quad \xrightarrow{ad_\Phi} \quad End(V) \otimes \Omega^1$$

When (E, Φ) is a Hodge bundle, $\mathcal{H}^1(K_\bullet)$ decomposes into weight spaces of the natural $\mathbb{C}^*$-action.

If there is a negative weight space, then the critical point is not a local minimum.

$U(p,1)$-Binary Hodge bundles

Assume $c = 0$.

Suffices to show B_{d_P,d_Q} is connected.

Canonical factorization:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f_1} & V_P & \xrightarrow{f_2} & V_2 & \longrightarrow & 0 \\ & & & & \Phi\downarrow & & \varphi\downarrow & & \\ & & & & V_Q \otimes \Omega & = & W_1 & & \end{array}$$

Semi-stability implies the bound on τ:

Theorem 5 $\tau \leq 2(g-1)$.

Let $d_2 = deg(V_2)$ and $B_{d_P,d_Q}(d_2) \subset B_{d_P,d_Q}$ the subset with the canonical factorization. B_{d_P,d_Q} is stratified by the $B_{d_P,d_Q}(d_2)$'s.

Families of Higgs bundles

Proposition 3 *B_{d_P,d_Q} is connected.*

Construct a family that parameterizes all possible V_1's in the canonical decomposition (semi-stability implies such family is bounded). The construction uses the Grothendieck Quot scheme.

Construct all possible V_2's and V_Q's. These are line bundles, so we are just dealing with Jacobi varieties. Care must be taken about the map ψ.

Construct all possible extension $Ext^1(V_2, V_1)$.

We thus end with a gigantic smooth object F that parameterize a family of Higgs bundles which contains all elements in $B_{d_P,d_Q}(d_2)$.

The condition of semi-stability is open. Hence contains an open and connected subscheme that maps onto $B_{d_P,d_Q}(d_2)$.

Deformation of $B_{d_P,d_Q}(d_2)$

$$B_\tau = \bigcup_{d_2} B_{d_P,d_Q}(d_2)$$

Fix a set of distinct points

$A = \{x_1, ..., x_{d_2}, y_1, ..., y_{d_2-1}, z_1, ..., z_{d_2-d-1}\} \subset X$

and let $Y = X \setminus A$. Fix $y \in Y$. For $t \in Y$, consider the following divisors on X:

$$D_2 = \sum_{i=1}^{d_2} x_i, \quad C(t) = D_2 - t - \sum_{i=1}^{d_2-1} y_i,$$

$$C = D_2 - y - \sum_{i=1}^{d_2-d-1} z_i.$$

Y parameterizes a family of Higgs bundles

$$V_P(t) = \mathcal{O}(C) \bigoplus_{i=1}^{p-1} \mathcal{O}(C_i(t)),$$

Deformation of $B_{d_P,d_Q}(d_2)$ (Continue)

Denote by p and $p_i(t)$, the projections to $\mathcal{O}(C)$ and $\mathcal{O}(C_i(t))$.

$D_2 - C(t)$ and $D_2 - C$ define maps $h_i(t)$: $\mathcal{O}(C_i(t)) \longrightarrow \mathcal{O}(D_2)$ and $h : \mathcal{O}(C) \longrightarrow \mathcal{O}(D_2)$. These maps induce a map

$$G_t : V_P(t) \longrightarrow \mathcal{O}(D_2), \quad G_t = h + \sum_{i=1}^{p-1} h_i(t).$$

Let $V_2 = \mathcal{O}(D_2)$. Since $d_2 \leq (2g-2)-d$, there exists $V_Q \in J^{-d}$ and $0 \not\equiv \varphi \in \mathcal{H}^0(V_2^{-1} \otimes V_Q \otimes \Omega)$. Let $\Phi(t) = \varphi \circ G_t$.

$(V_P(t) \oplus V_Q, \Phi(t))$ is a family of Higgs bundles parameterized by Y. Let p_P, p_Q be the projections onto the $V_P(t), V_Q$ factors.

Proposition 4 *The Higgs bundle* $(V_P(t) \oplus V_Q, \Phi(t))$ *is in* $B_{d_P,d_Q}(d_2 - 1)$ *if* $t = y$ *and in* $B_\tau(d_2)$ *if* $t \neq y$.

$\underline{U(p,p)\text{-binary bundles}}$

Canonical factorization:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \xrightarrow{f_1} & V_P & \xrightarrow{f_2} & V_2 & \longrightarrow & 0 \\ & & & & \Phi_1\downarrow & & \varphi\downarrow & & \\ 0 & \longleftarrow & W_2 & \longleftarrow & V_Q \otimes \Omega & \longleftarrow & W_1 & \longleftarrow & 0 \end{array}$$

Semi-stability implies the bound on τ:

Theorem 6 $\tau \leq 2p(g-1)$.

Lemma 1 *Suppose* $\tau > 2(p-1)(g-1)$. *Then* $(V_P \oplus V_Q, (\Phi_1, \Phi_2)) \in M_{d_P, d_Q}$ *implies*

$$\Phi_1 : V_P \longrightarrow V_Q \otimes \Omega$$

is generically surjective.

Proof: Otherwise the sub-Higgs bundle $(V \oplus W_1 \otimes \Omega^{-1}, (\Phi_1, \Phi_2))$ is Φ-invariant and with a larger slope.

Families of Higgs bundles

Proposition 5 *B_{d_P,d_Q} is irreducible.*

Construct a family that parameterizes all possible V_Q's in the canonical decomposition (semistability implies such family is bounded). Construct

$$Q_1 := Quot^H_{\oplus_{i=1}^{H(1)} L^{-1}/X/\mathbb{C}},$$

the Grothendieck scheme parameterizing the quotient sheaves of $\oplus_{i=1}^{H(1)} L^{-1}$ with Hilbert polynomial $H(m) := \chi(V_Q \otimes L^m)$, where L is some ample line bundle.

Q_1 contains an irreducible and smooth quasi-projective variety

$R = \{W \in Q_1 : W$ is locally free and $\mathcal{H}^1(W) = 0\}$

Families of Higgs bundles

Construct

$$E \longrightarrow X \times R$$

the universal quotient bundle of $\oplus_{i=1}^{H(1)} L^{-1}$. Then there exists a relative Quot scheme

$$Q_2 \ := \ Quot^{-d_P+d_Q+2p(g-1)}_{E/X\times R/R}$$

parameterizing quotient sheaves of E supported as length $-d_P + d_Q + 2p(g-1)$ subschemes of a fiber of $X \times R \to R$.

We thus end with a gigantic smooth object Q_2 that parameterize a family of Higgs bundles which contains all elements in B_{d_P,d_Q}.

Proposition 6 *Q_2 is irreducible.*

The condition of semi-stability is open. Hence contains an open and connected subscheme that maps onto B_{d_P,d_Q}.

Irrducibility of Q_2

$Q_2 \to R$ factors through a surjective morphism

$$h \; : \; Q_2 \quad \to \quad R \times X^{(-d_P+d_Q+2p(g-1))},$$

$Q_2^{free} \subset Q_2$ parameterizing pairs of quotient sheaves $(\oplus_{i=1}^{H(1)} L^{-1} \to W, W \to F)$, where F is supported on a subscheme $D \subset X$ as a free $\mathcal{O}_D$-module of rank 1.

The restriction of h to Q_2^{free} is a smooth.

R, $X^{(-d_P+d_Q+2p(g-1))}$ and the fibers of h (restricting to Q_2^{free}) are irreducible.

Each fiber of h in Q_2^{free} is dense in the fiber in Q_2

Lemma 2 *The Quot scheme $Q(\ell, \mathcal{O}^p_{(x)})$, of length l quotients of the stalk at x of the trivial rank p vector bundle, is irreducible.*

m the maximal ideal of x, $A = \mathcal{O}_{(x)}/m^\ell$. $G = GL(p, A)$

Any quotient sheaf in $Q(\ell, \mathcal{O}^p_{(x)})$ is also a quotient of the free A-module of rank p.

Deformation: $\eta(t) := \psi + t\varphi$, $t \in \mathbb{C}$,

$$\psi = \begin{pmatrix} 0 & & & \cdots & 0 & z^{\ell_p} \\ z^{\ell_1} & 0 & & \cdots & & 0 \\ 0 & z^{\ell_2} & 0 & \cdots & & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & 0 & z^{\ell_{p-1}} & 0 \end{pmatrix} \quad \text{and}$$

$$\varphi = \begin{pmatrix} 1 & 0 & \cdots & & 0 & 0 \\ 0 & 1 & & & 0 & \\ & 0 & & & \vdots & \vdots \\ \vdots & \vdots & & 1 & 0 & \\ & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & z^\ell \end{pmatrix}.$$

$$\eta(t)_p = \frac{z^{\ell_p}}{t}\eta(t)_1 - \sum_{k=2}^{p-1}\left[\left(\frac{-1}{t}\right)^k z^{(\ell_p + \sum_{i=1}^{k-1}\ell_i)}\right]\eta(t)_k$$

$$+ \left[t + \left(\frac{-1}{t}\right)^{p-1}\right] \cdot z^\ell \cdot e_p.$$

Some related results

- [Hitchin]The components in $Hom(\pi(X), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$ are vector bundles of rank $e+(g-1)$ over the $2(g-1)-e$ symmetric products of X, where e is the Euler characteristic (invariant).

- [Xia] The components in $Hom(\pi(X), SO(2,1))/SO(2,1)$ that do not correspond to components in $Hom(\pi(X), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$ are either the total space of a vector bundle of rank $2g-2$ over a torus of complex dimension $g-1$ or contain an open dense set that is a vector bundle of rank $2g-2$ over a torus of complex dimension $g-1$ with the identity removed.

Some related results (continue)

- [Xia] $Hom(\pi(X), PU(2,1))/PU(2,1)$ has $6(g-1)+1$ connected components, and these components are distinguished by the Toledo invariants.

- [Gothen] The Betti numbers of the smooth components of $Hom(\pi(X), PU(2,1))/PU(2,1)$ have been computed.

- [Bradlow, Garcia-Prada, Gothen] Each component of $Hom(\pi_1(X), PU(p,q))/PU(p,q)$, associated with a fixed τ and a fixed Chern class c, is connected.