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The gerbe of Higgs bundles

X - smooth compact curve
G - reductive group (e.g. GL_n)

Can consider $\mathrm{Bun}_G X$ = moduli space / stack of sstable / all G-bundles on X

There is a Hitchin map

$$\begin{array}{ccc} T^* \mathrm{Bun}_G X & \xrightarrow{H} & B_G(X, K) := \bigoplus_{i=1}^{r} H^0(X, K^{\otimes d_i}) \\ \cap & & \uparrow \\ & & \text{Hitchin base} \\ \mathcal{H}iggs_G(X, K) & & \end{array}$$

Here
K = canonical bundle of X
r = rank of G
$\{d_i\}$ = degrees of the basic invariant polynomials of G.

$\mathcal{H}iggs_G(X,K)$ = moduli of pairs $(\mathcal{P}, s \in \Gamma(X, \mathrm{ad}\mathcal{P} \otimes K))$ where $\mathcal{P}$ - principal G-bundle on X.

$H(\mathcal{P}, s)$:= invariant polynomials of s

Remarks: • if we work with the stacks $\Rightarrow$
$\mathcal{H}iggs_G(X, K) = T^* Bun_G(X)$.

• the fibers of H are Jacobians, Pryms etc.

Goals: • extend the story to all G (replace spectral data by cameral data)
• extend to all X (any scheme $/\mathbb{C}$)
• extend the story to all K (e.g. any line bundle or any abelian group scheme)
• remove the explicit dependence on K
i.e. replace the fibration
$$\begin{array}{c} \mathcal{H}iggs_G(X, K) \\ \downarrow \\ B_G(X, K) \end{array}$$
by a universal (K-independent) fibration

$$\begin{array}{cl} \mathcal{H}iggs_G(X) & \text{- abstract Higgs bundles} \\ \downarrow & \\ B_G(X) & \text{- moduli of abstract cameral covers} \end{array}$$

Then we will have a fiber product

$$\begin{array}{ccc} \mathcal{H}iggs_G(X, K) & \longrightarrow & \mathcal{H}iggs_G(X) \\ \downarrow & & \downarrow \\ B_G(X, K) & \longrightarrow & B_G(X) \end{array}$$

coming from assigning values in K to our abstract Higgs data.

- Interesting cases: K - line bundle
  K - multiplicative group scheme (Sklyanin)
  $Y \to X$ - an elliptic fibration

- precise description of fibers - gerbe twisting.

- K - valued Higgs bundles
- Bundles on elliptic fibrations
- Bundles on genus one fibrations
- SYZ conjecture with a B-field
- Duality of Hitchin systems
- del Pezzo fibrations vs E_8 bundles.

Recall $g \in G$ is called regular if
$$\dim Z_G(g) = \dim Z_{\mathfrak{g}}(g) = r$$

A subgroup in G or a subalgebra in $\mathfrak{g}$ is called a <u>regular centralizer</u> if it is of the form $Z_G(g)$ (respectively $Z_{\mathfrak{g}}(g)$) for some regular element $g \in G$.

Let $N := N_G(T)$ for $T \subset G$ a fixed maximal torus

Have a ses of groups

$$1 \to T \to N \to W \to 1$$

with W = Weyl group of G.

Consider the following basic diagram

$$\begin{tikzcd} G/T \arrow[r, hook] \arrow[d] & \overline{G/T} \arrow[d] \\ G/N \arrow[r, hook] & \overline{G/N} \end{tikzcd}$$

space of maximal tori in G

space of regular centralizers in G.

<u>Example:</u> $G = GL(2) \Rightarrow$ the basic diagram is

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^1 - \text{diagonal} \arrow[r, hook] \arrow[d] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "2:1 \text{ cover}"] \\ \mathbb{P}^2 - \text{discriminant} \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd}$$

<u>Def:</u> Given a scheme X and a group G define a G-Higgs bundle on X as a pair $(\mathcal{P}, \underline{C})$ where:

- $\mathcal{P} \to X$ is a principal G-bundle on X
- $\underline{C} \subset \operatorname{ad} \mathcal{P}$ is a subbundle of regular centralizers

Equivalently a G-Higgs bundle on X is a pair $(\mathcal{P}, \sigma)$ where $\mathcal{P}$ is a principal G-bundle and

$$\sigma : \mathcal{P} \to \overline{G/N}$$

is a G-equivariant map.

<u>Def:</u> A cameral cover of X is a W-Galois cover $\tilde{X} \to X$ which locally is a pullback of the standard cover.

Here the standard cover is either

$$\underline{t} \to \underline{t}/W$$

or

$$\overline{G/T} \to \overline{G/N}$$

(These two local models lead to the same notion of a cameral cover since we have a diagram

$$\begin{tikzcd}
\overline{G/T} \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\mathfrak{g}}_{reg} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{t} \arrow[d] \\
\overline{G/N} & \mathfrak{g}_{reg} \arrow[l] \arrow[r] & \underline{t}/W
\end{tikzcd}$$

where both squares are cartesian).

<u>Proposition</u>: When $G = GL_n \Rightarrow$ the categories of cameral covers and of spectral covers (for the fundamental representation) are equivalent.

<u>Relation to the usual story</u>: If K - line bundle $\Rightarrow$ a <u>K-valued Higgs bundle</u> is a pair $(\mathcal{P}, s \in \Gamma(X, \mathrm{ad}\,\mathcal{P} \otimes K))$.

A <u>regular K-valued Higgs bundle</u> is a pair $(\mathcal{P}, s)$ s.t. $s(x)$ - regular for all $x \in X$.

A <u>regularized K-valued Higgs</u> is a triple

$$(\underline{P}, \underline{c}, s)$$

s.t.:

- $(\underline{P}, \underline{c})$ is an abstract Higgs bundle
- $s \in \Gamma(X, \underline{c} \otimes K) \subset \Gamma(X, \mathrm{ad}\, \underline{P} \otimes K)$

Now consider $\mathcal{H}iggs_G(X)$ – the stack of abstract G-Higgs bundles

$\mathcal{C}am_G(X)$ – ~~the stack of~~ cameral covers of X

There is an abstract Hitchin map

$$H : \mathcal{H}iggs_G(X) \to \mathcal{C}am_G(X)$$

Explicitly $H((\underline{P}, \sigma : \underline{P} \to \overline{G/N}))$ is defined as the descent of the cover

$$\sigma^*(\overline{G/T}) \to \underline{P}$$

to X (one has to show that this descent exists)

The next task is to define the fibers of H.

There is a natural sheaf of commutative groups

$$\overline{T}_{\tilde{X}} \quad \text{on} \quad X$$

depending on a given cameral cover

$$\tilde{X} \to X.$$

By definition $\overline{T}_{\tilde{X}}(U) = \mathrm{Maps}_W(\tilde{U}, T)$.

This sheaf can be corrected as follows. Define

$$T_{\tilde{X}} \subset \overline{T}_{\tilde{X}}$$

where

$$T_{\tilde{X}}(U) := \left\{ t \in \mathrm{Maps}_W(\tilde{U}, T) \,\middle|\, \begin{array}{l} \forall\ \alpha\text{-root} \\ \Rightarrow \alpha \circ t_{|D^{\alpha}_{\tilde{X}}} = +1 \end{array} \right\}$$

Here $D^{\alpha}_{\tilde{X}}$ is the α-component of the discriminant of $\tilde{X} \to X$.

With this notation we now have

<u>Thm (Donagi–Gaitsgory)</u> $\mathrm{Higgs}_{\tilde{X}}(X)$:= fiber of $H : \mathcal{H}iggs_G(X) \to \mathrm{Cam}_G(X)$ over $(\tilde{X} \to X)$ is a $H^1(X, T_{\tilde{X}})$-torser. More precisely $\mathrm{Higgs}_{\tilde{X}}$ is a gerbe bound by $\mathrm{Tors}_{T_{\tilde{X}}}$

One can identify the gerbe $\mathrm{Higgs}_{\tilde{X}}(X)$ explicitly. This involves the following 3 ingredients

- a twist along the ramification of $\tilde{X} \to X$
- a twist by the class of the extension
$$[1 \to T \to N \to W \to 1] \in H^2(W, T)$$
- a twist by non-primitive coroots (this appears only if G has a direct factor of type $SO(2n+1)$).

<u>Applications:</u>

- K-valued Higgs bundles

<u>Thm</u> Regularized K-valued G-Higgs bundles /X are equivalent to 3 pieces of data

(1) a Cameral cover $\tilde{X} \to X$
(2) a W-equivariant map over X:
$$v : \tilde{X} \to \underline{t} \otimes K$$
(3) an object of $\mathrm{Higgs}_{\tilde{X}}(X)$.

<u>Corrolary</u>: Regular K-valued Higgs bundles on X correspond to
(1) — " —
(2) with v - an embedding
(3) — " —

. $Y \to X$ elliptic fibration with integral fibers

A regularized G bundle on Y is a G-bundle + a reduction of the structure group to some family of regular centralizers

<u>Thm</u>: Regularized G-bundles on $Y \longleftrightarrow$
(1) Cameral $\tilde{X} \to X$
(2) W-equivariant map
$$v : \tilde{X} \to \mathrm{Bun}_T(Y/X)$$
+ consistency condition on ramification gerbe.
(3) an object of $\mathrm{Higgs}_{\tilde{X}}(X) \underset{\mathrm{Tors}\, T_{\tilde{X}}}{\otimes} Q_v$.