

The gerbe of Higgs bundles

X - smooth compact curve

G - reductive group (e.g. GL_n)

Can consider $Bun_G X = \text{moduli space of stable } G\text{-bundles on } X$

There is a Hitchin map

$$\begin{array}{ccc}
 T^* Bun_G X & \xrightarrow{H} & B_G(X, K) := \bigoplus_{i=1}^r H^0(X, K^{\otimes d_i}) \\
 \cap & & \uparrow \\
 & & \text{Hitchin base} \\
 & & \text{Higgs}_G(X, K)
 \end{array}$$

Here K = canonical bundle of X

r = rank of G

$\{d_i\}$ = degrees of the basic invariant polynomials of G .

$\text{Higgs}_G(X, K) = \text{moduli of pairs } (E, s \in \Gamma(X, \text{ad} E \otimes K))$
where E - principal G -bundle on X .

$H(E, s) = \text{invariant polynomials of } s$

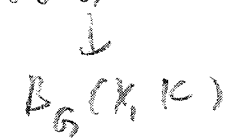
Remarks:

- if we work with the stacks \Rightarrow
 $\text{Higgs}_G(X, k) = T^* \text{Bun}_G(X)$.
- the fibers of H are, Jacobians, Pryms etc.

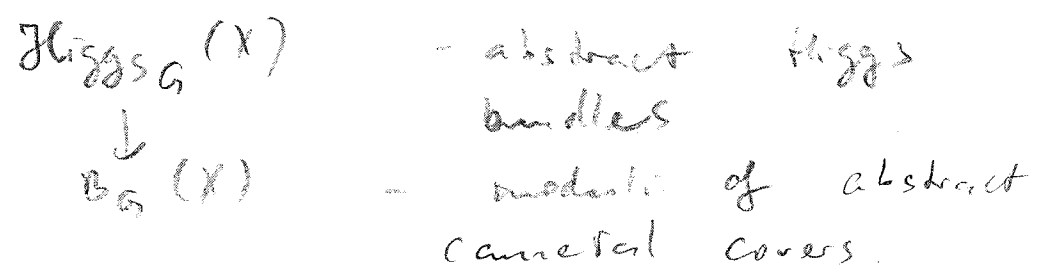
Goals:

- extend the story to all G (replace Spectral data by Cameral data)
- extend to all X (any scheme / \mathbb{C})
- extend the story to all k (e.g. any base bundle or any abelian group scheme)
- Remove the explicit dependence on k

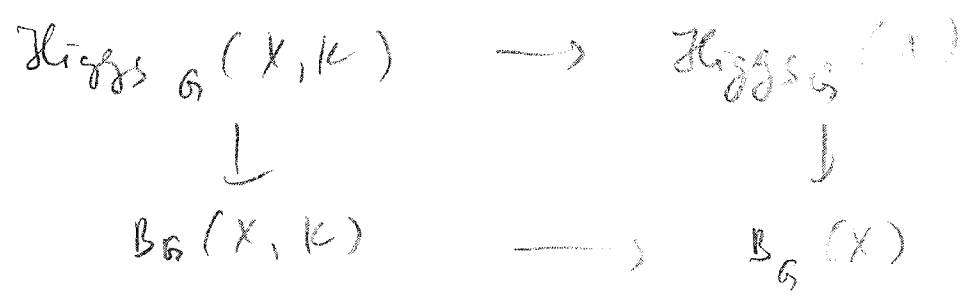
i.e. replace the fibration $\text{Higgs}_G(X, k)$



by a universal (k -independent) fibration



Then we will have a fiber product



coming from assigning values in K to
out abstract Higgs data.

- Interesting cases: K -line bundle
 K -multiplicative group
 scheme (Selyagin)
 $Y \rightarrow X$ - an elliptic fibration
 - precise description of fibers - gerbe
 twisting.
 - K -valued Higgs bundles
 - Bundles on elliptic fibrations
 - Bundles on genus one fibrations
 - SYZ conjecture with a \mathbb{R} -field
 - Duality of Hitchin systems
 - del Pezzo fibrations vs E_6 bundles.
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Recall $g \in G$ is called regular if
 $\dim Z_G(g) = \dim Z_{\mathfrak{g}}(g) = r$

A subgroup in G or a subalgebra
 in \mathfrak{g} is called a regular centralizer
 if it is of the form $Z_G(g)$ (respectively
 $Z_{\mathfrak{g}}(g)$) for some regular element $g \in G$.

Let $N := N_G(T)$ for $T \subset G$
 a fixed maximal torus
 Have a seq. of groups

$$1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1$$

with $W = \text{Weyl group of } G$.

Consider the following basic diagram

$$\begin{array}{ccc}
 G/T & \hookrightarrow & \overline{G/T} \\
 \downarrow & & \downarrow \\
 G/N & \hookrightarrow & \overline{G/N} \\
 \uparrow & & \uparrow \\
 \text{space of} & & \text{space of} \\
 \text{maximal tori} & & \text{regular} \\
 \text{in } G & & \text{centralizers in } G.
 \end{array}$$

Example: $G = GL(2) \Rightarrow$ the basic diagram is

$$\begin{array}{ccc}
 \mathbb{P}^1 \times \mathbb{P}^1 - \text{diagonal} & \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\
 \downarrow & & \downarrow \text{2:1 cover} \\
 \mathbb{P}^1 - \text{diagonal} & \hookrightarrow & \mathbb{P}^1
 \end{array}$$

Def: Given a scheme X and a group G define a G -Higgs bundle on X as a pair (P, \underline{C}) where:

- $P \rightarrow X$ is a principal G -bundle on X
- $\underline{C} \subset \text{ad } P$ is a subbundle of regular centralizers

Equivalently a G -Higgs bundle on X is a pair (P, σ) where P is a principal G -bundle and

$$\sigma: P \rightarrow \overline{G/N}$$

is a G -equivariant map.

Def: A canonical cover of X is a W -Galois cover $\tilde{X} \rightarrow X$ which locally is a pullback of the standard cover.

Here the standard cover is either

$$\mathbb{t} \rightarrow \mathbb{t}/W$$

or

$$\overline{G/T} \rightarrow \overline{G/N}$$

(These two local models lead to the same notion of a central cover since we have a diagram

$$\begin{array}{ccccc}
 \overline{G/T} & \swarrow & \tilde{g}_{\text{reg}} & \searrow & \pm \\
 & \square & \downarrow & \square & \downarrow \\
 \overline{G/N} & \swarrow & g_{\text{reg}} & \searrow & \pm/W
 \end{array}$$

where both squares are cartesian).

Proposition: When $G = GL_n \Rightarrow$ the categories of central covers and of spectral covers (for the fundamental representation) are equivalent.

Relation to the usual story: If K -line bundle \Rightarrow a K -valued Higgs bundle is a pair $(P, s \in \Gamma(X, \text{ad } P \otimes K))$.

A regular K -valued Higgs bundle is a pair (P, s) s.t. $s(x)$ - regular for all $x \in X$.

A regularized K -valued Higgs is

a triple

$$(P, \underline{E}, s)$$

with

- (P, \underline{E}) is an abstract Higgs bundle
- $s \in \Gamma(X, \underline{E} \otimes K) \subset \Gamma(X, \text{ad } P \otimes K)$

Now consider $\mathcal{Higgs}_G(X)$ - the stack of abstract G -Higgs bundles

$\text{Cam}_G(X)$ - the stack of Cameral covers of X

There is an abstract Hitchin map

$$H: \mathcal{Higgs}_G(X) \rightarrow \text{Cam}_G(X)$$

Explicitly $H((P, \sigma: P \rightarrow \overline{G/T}))$ is defined as the descent of the cover

$$\sigma^* (\overline{G/T}) \rightarrow P$$

to X (one has to show that this descent exists)

The next task is to define the fibers of H ,

There is a natural sheaf of commutative groups

depending on a given central cover $\tilde{X} \rightarrow X$

By definition $\overline{T}_{\tilde{X}}(\tilde{U}) = \text{Maps}_{\mathbb{W}}(\tilde{U}, T)$.

This sheaf can be corrected as follows. Define

$$T_{\tilde{X}} \subset \overline{T}_{\tilde{X}}$$

where

$$T_{\tilde{X}}(\tilde{U}) := \left\{ t \in \text{Maps}_{\mathbb{W}}(\tilde{U}, T) \mid \begin{array}{l} \text{# } d\text{-root} \\ \Rightarrow \text{dot} / \mathcal{D}_{\tilde{X}}^d = +1 \end{array} \right\}$$

Here $\mathcal{D}_{\tilde{X}}^d$ is the d -component of the discriminant of $\tilde{X} \rightarrow X$.

With this notation we now have

Thm (Donagi-Gaiitsgory) $\text{Higgs}_X(X) :=$ fiber
of $H : \text{Higgs}_G(X) \rightarrow \text{Conn}_G(X)$ over
 $(\tilde{X} \rightarrow X)$ is a $H^1(X, T_{\tilde{X}})$ -torsor.
More precisely Higgs_X is a gerbe
bound by $\text{Tors}_{T_{\tilde{X}}}$

One can identify the gerbe $\text{Higgs}_X(X)$
explicitly. This involves the following
3 ingredients

- a twist along the ramification
of $\tilde{X} \rightarrow X$
- a twist by the class of
the extension

$$[1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1] \in H^2(W, T)$$

• a twist by non-primitive
cocycles (this appears only if
 G has a direct factor of type
 $SO(dn+1)$).

Applications:

- k -valued Higgs bundles

Thm: Regularized k -valued G -Higgs bundles (X)
are equivalent to 3 pieces of data

- (1) a Cameral cover $\tilde{X} \rightarrow X$
- (2) a W -equivariant map over X :

$$v: \tilde{X} \rightarrow \underline{t} \otimes K$$
- (3) an object of $\text{Higgs}_{\tilde{X}}(X)$.

Corollary: Regular K -valued Higgs bundles on X correspond to

- (1) — " —
- (2) with v - an embedding
- (3) — " —

• $Y \rightarrow X$ elliptic fibration with integral fibers

A regularized G bundle on Y is a G -bundle + a reduction of the structure group to some family of regular centralizers

Thm: Regularized G -bundles on $Y \iff$

- (1) Cameral $\tilde{X} \rightarrow X$
- (2) W -equivariant map

$$v: \tilde{X} \rightarrow \text{Bun}_T(Y/X)$$

+ consistency condition on ramification gerbe.

- (3) an object of $\text{Higgs}_{\tilde{X}}(X) \otimes_{\text{Tor}_2^{\mathbb{Z}}} Q_V$.