Group valued Higgs bundles over a punctured elliptic curve

Joint work with J. Hurtubise.

math.AG/0203031

Results:

1) $M(G,c)$ has a holomorphic Poisson structure $\Psi \in \Lambda^2 TM(G,c)$

2) The symplectic leaves are classif[ied] by labeling finitely many pts of Σ, each by a DOMINANT COCHARACTER of $T \subset G$.

3) $\exists$ an analogue of the Hitchin map.

4) The symp. leaves are finite dim'l and A.C.I.H.S

Plan of the Talk:

1) Motivation: $M(G,c)$ is an analogue of a loop group.
   a) Quantum (loop) groups, Poisson gps and the C.Y.B.E
   b) C.F.T. and the CDYBE.

2) The symp. leaves $M(G,c,\mathcal{D})$ of $M(G,c)$

3) The symp. structure on " via defor. T[h]

4) Complete Integrability of $M(G,c,\mathcal{D})$

Introduction:

Let Σ be an elliptic curve,
G a complex reductive group.

GOAL:

Describe the algebraic and symplectic geometry of the moduli space (Ind variety)

$$M(G,c) := \{ (\underline{P}, \varepsilon) : \underbrace{\qquad\qquad}_{\substack{\underline{P} \text{ is a principal } G\text{-bundle} \\ \text{of topological type } c \in H^2(\Sigma, \pi_1(G)) \\ \varepsilon \text{ a meromorphic section} \\ \text{of the adjoint GROUP bundle } \underline{P}(G)}} \} / \sim$$

Note: If $G = GL(n)$, then $(\underline{P}, \varepsilon)$ is a meromorphic Higgs pair because $GL(n) \subset \mathfrak{gl}_n$

CYBE

$\mathfrak{g}$ a simple Lie algebra / $\mathbb{C}$

$r : \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$

$\mathfrak{g} \subset A$ an embedding in an associative algebra ($\mathcal{U}(\mathfrak{g})$, ...)

$$\boxed{\left[r^{12}(z_1-z_2), r^{13}(z_1-z_3)\right] + \left[r^{12}(z_1-z_2), r^{23}(z_2-z_3)\right] + \left[r^{13}(z_1-z_3), r^{23}(z_2-z_3)\right] = 0}$$

Solutions satisfying, in addition

a) $r(z) = -r^{21}(-z)$

b) Residue at 0 (which is symmetric) is G-inv,

are used to construct Poisson structures on LG, such that the multiplication

$$m : LG \times LG \to LG$$

is a Poisson map.

Solutions of CYBE are classified [BD].

∃ Rational, Trigonomometric, and Elliptic.

$$\underbrace{\textbf{and Elliptic}}_{\textbf{ONLY FOR TYPE } A_n}$$

The open Question:

Certain symp leaves of the elliptic quadratic Poisson str on the type A loop group are ACIHS (Sklyanin System).

Our Goal:

Find the analogous ACIHS for any reductive G.

The phase space as a relative loop gp

Continuous loop gp: LG_{cont}

$$S^1 = \bigcirc \xrightarrow[\text{continuous}]{\gamma} G$$

Analytic loop gp: LG_{an}

Embed $S^1 \subset \mathbb{C}$. Consider loops admitting a holomorphic extension to an annulus:

 $\xrightarrow{\gamma} G$

Meromorphic loop gp: LG_Σ

Embed $S^1 \subset \Sigma$.

LG_Σ = mero function from Σ to G.

A twisted analogue:

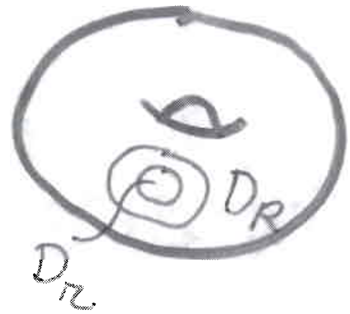
Fix a principal G-bundle $\underline{P} \to \Sigma$.

Let $\underline{P}(G)$ be the ADJOINT GROUP bundle.

Consider meromorphic sections $\gamma : \Sigma \to P(G)$

Advantage: If $\underline{P}$ is RIGID, we get a FACTORIZATION of the loop algebra, hence a local factorization of LG.

Factorization: Embed the annulus

$A = D_R \setminus \bar{D}_r \subset$  $= \Sigma$

$$0 \to \overbrace{H^0(\Sigma, P\mathfrak{g})}^{0 \text{ by S.D.}} \to \underbrace{\Gamma(D_R, P\mathfrak{g})}_{L\mathfrak{g}_+} \oplus \Gamma(\Sigma \setminus \bar{D}_r, P\mathfrak{g}) \xrightarrow{\sim} \overbrace{\Gamma(A, P\mathfrak{g})}^{L\mathfrak{g}}$$

$$\to \underbrace{H^1(\Sigma, P\mathfrak{g})}_{0 \text{ by rigidity}} \to 0$$

Thm: Over an elliptic curve Σ, the only rigid bundles are the stabl bundles corresponding to (G, c), whe $G = PGL(n)$ and c is a generator of $\pi_1(PGL(n)) \simeq \mathbb{Z}_n$

Relation with the CYBE:

when P is rigid, the solution

$r : \mathbb{C} \longrightarrow End(\mathfrak{g})$ of the CYBE

is the KERNEL of an integral formul for the projection on $L\mathfrak{g}_+$

$\forall z' \in D_R, \quad \int_{\partial D_R} r(z - z')\phi(z)\,dz = \begin{cases} \phi(z') & \text{if } \phi \in \\ 0 & \text{if } \phi \in \end{cases}$

Motivation: (cont.)

Ideas from C.F.T led Felder to discover a generalization of the CYBE: The CDYBE.

- It is a diff eq!
- Etingof–Varchenko: Poisson str on

$$\begin{array}{c} V \times LG \times V \\ {\scriptstyle \text{open}}\cap \\ \mathfrak{t} \end{array}$$

- Our Algebro-geometric interpretation:
  - $\operatorname{Pic}^{\circ}\Sigma = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$
  - Moduli of semi-stable T-bundles of triv top type $= \mathfrak{t}_{\mathbb{C}} / [Ch(T)^{*} \oplus \tau \cdot ch(T)^{*}]$
  - Every Semi-stable G-bundle, of trivial top type, is S-equiv to one, which is induced from a T-bundle
  - $Bun_{G} = Bun_{T}/W = \mathfrak{t}_{\mathbb{C}} / W_{\text{aff}}$.
  - Reduction: $\Delta(\mathfrak{t}) \times LG / T$

    is "essentially" our $M(G, \circ)$.

Singularity Types:

Denote by $G[[t]]$ the group of formal germs at $0 \in \Delta$ of holomorphic maps from Δ to G.

Let $G((t))$ be the formal analogue for the punctured disk $\Delta \setminus \{0\}$.

$G[[t]] \times G[[t]]$ acts on $G((t))$ via left and right multiplication.

Iwahori's Thm: $G((t)) = G[[t]]\, T((t))\, G[[t]]$

So, a double orbit $\mathbb{O}$ is determined by $\mathbb{O} \cap T((t))$, which is a $W \times T[[t]]$ orbit.

Cor: There is a 1-1 correspondence between

1) $G[[t]] \times G[[t]]$ orbits in $G((t))$.
2) $G[[t]]$ orbits in the loop grassmanian $G((t))/G[[t]]$
3) W-orbits in the co-character lattice $ch(T)^* = Hom(\mathbb{C}^\times, T)$.

$$\underset{\substack{\Uparrow \\ T((t))}}{G[[t]]\, \gamma\, G[[t]]} \longleftrightarrow Ord_0(\ ,\gamma) \in Ch(T$$

$: ch(T) \to$
$\lambda \longmapsto Ord_0(\lambda$

Def: A singularity Data $\mathbb{O}$ is a labeling of finitely many points $P_1, P_2, \ldots, P_d$ in Σ by W-orbits in $Ch(T)^*$ $\mathbb{O}_{P_1}, \ldots, \mathbb{O}_{P_d}$.

Let $M_\Sigma(G, c, \mathbb{O}) \subset M_\Sigma(G, c)$

$\overset{\text{def}}{=}$

The moduli space of isomorphism classes of simple pairs $(\underline{P}, \varepsilon)$, of a principal G-bundle $\underline{P}$ of topological type c, and a meromorphic section ε with singularities in $\mathbb{O}$.

Thm: [HM]

1) $M_\Sigma(G, c, \mathbb{O})$ are the symplectic leaves of $M_\Sigma(G, c)$. Each is smooth, symplectic, and finite dimensional.

$$\boxed{\dim M_\Sigma(G, c, \mathbb{O}) = 2\dim(\mathfrak{z}) + \sum_{p \in \Sigma} \dim(\underbrace{\mathbb{O}_p}_{\cap\, G((t))/G[[t]]})}$$

2) $M_\Sigma(G, c, \mathbb{O})$ is an ACIHS.

Examples:

0) $G=\mathbb{C}^\times$, $\mathbb{D}$ empty: $\mathcal{M}(\mathbb{C}^\times, c, \phi) = \Sigma \times \mathbb{C}^\times$

$$\Psi = dz \wedge \frac{d\lambda}{\lambda}$$

1) Ruijsenaars–Schneider systems (deformations of the type A Calogero–Moser):

$G = PGL(n,\mathbb{C})$, $c \in \pi_1(G)$ a generator,

$$\mathbb{D} = \mathbb{D}_{p_1} + \mathbb{D}_{p_2}$$

$$\mathbb{D}_{p_1} \ni \begin{bmatrix} t & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots 1 \end{bmatrix} \qquad \mathbb{D}_{p_2} \ni \begin{bmatrix} t^{-1} & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$\mathcal{M}(G, c, \mathbb{D})$ is birational to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

2) Products of Grassmannians.

3)

The tangent space at $(\underline{P}, t) \in M_\Sigma(G, c, \mathbb{O})$

1) 1^{st} order deformations of $\underline{P}$: $H^1(\Sigma, \underline{P}(\mathfrak{g}))$

2) Deformations of a holomorphic $t : \Sigma \to P(G)$ are given by sections of $ad(P, t) = N_{t(\Sigma)}$

Observe: $R_t, L_t : \underline{P}\mathfrak{g} \xrightarrow{\cong} ad(P, t)$

3) Fix $\underline{P}$ and a singularity type $\mathbb{O}$. Deformations of t should stay in the image of

$$\underline{P}(G) \times P(G) \xrightarrow{m} P(G)$$
$$(a \ , \ b) \longmapsto a t b.$$

Infinitesimally:

Definition: $ad(P, t) := \underline{P}\mathfrak{g} \oplus P\mathfrak{g} / Ker(dm)$

$$\boxed{0 \to \underbrace{Ker(dm)}_{\{(a,b) \,:\, Ad_t(a) + b = 0\}} \to P\mathfrak{g} \oplus P\mathfrak{g} \xrightarrow{dm} ad(P, t) \to 0} \quad (*)$$

4) Deformations of (P, t) in $M_\Sigma(G, c, \mathbb{O})$: Given by the first hyper-cohomology of

$$\underline{P}\mathfrak{g} \xrightarrow{L_t - R_t} ad(\underline{P}, t)$$

The symplectic structure is induced by:

$$\begin{array}{ccc} \uparrow {\scriptstyle -R_t^*} & & \uparrow {\scriptstyle L_t} \\ ad(P, t)^* & \longrightarrow & P\mathfrak{g}^* \end{array}$$

The Jacobi identity:

Hard!

- Choose a faithful rep $G \hookrightarrow GL_N$
- $$M(G, c, \mathbb{D}) \hookrightarrow \underbrace{M(GL_N, \rho(c), \rho(\mathbb{D}))}_{\substack{\text{moduli of sheaves} \\ \text{on } \mathbb{P}[\omega_\Sigma(D) \oplus \mathcal{O}]}}$$

, Ψ is $\mathbb{C}^\times$ inv!

The dimension formula:

We need to calculate

$$\dim \mathbb{H}^1\left[P\mathfrak{g} \to \mathrm{ad}(P,\varepsilon)\right] =$$

$$\underbrace{\dim \mathbb{H}^0}_{\substack{\| \\ \dim \mathfrak{z}}} + \underbrace{\dim \mathbb{H}^2}_{\substack{\| \\ \dim \mathfrak{z}}} - \underbrace{\chi\left[P\mathfrak{g} \to \mathrm{ad}(P,\varepsilon)\right]}_{\chi(P\mathfrak{g}) - \chi\, \mathrm{ad}(P,\varepsilon)}$$

(because $\mathrm{ad}(P,\varepsilon) \simeq \mathrm{Ker}(dm)^*$)

$\chi(P\mathfrak{g}) = 0$, $\quad \underbrace{\chi\, \mathrm{ad}(P,\varepsilon)}_{\substack{\| \\ \chi\, \mathrm{Ker}(dm)}}$

$$= 2\dim \mathfrak{z} - \chi\, \mathrm{Ker}(dm)$$

Lemma: $-\deg \mathrm{Ker}(dm) \doteq \sum_{p\in\Sigma} \dim \mathbb{D}_p$

$\mathrm{Ker}(d_\varepsilon m) \subset P\mathfrak{g}$ via first proj

$$\| $$

$$\{a \in P\mathfrak{g} : \mathrm{Ad}_\varepsilon(a) \in P\mathfrak{g}\}$$

This reduces the calculation of $\deg \mathrm{Ker}(d_\varepsilon m)$ to a local (formal) calculation at the stalks:

Conceptually: The completed stalk of $\widehat{\mathrm{Ker}(d_\varepsilon m)}_{(p)}$ is the Lie alg of $\mathrm{Stab}_{G[[t]]}(\bar\varepsilon)$ of $\bar\varepsilon := \varepsilon\, G[[t]]$ in the loop Grassmannian $G((t))/G[[t]]$.

In $G((t))$: $\quad \mathrm{Stab}_{G((t))}\, \bar\varepsilon \xrightarrow{\mathrm{Ad}_\varepsilon^{-1}} \mathrm{Stab}_{G((t))}\, \bar 1 = G[[t]]$

In $G[[t]]$: $\quad \boxed{\mathrm{Stab}_{G[[t]]}\, \bar\varepsilon = G[[t]] \cap \mathrm{Ad}_\varepsilon\, G[[t]].}$

Thus, $\quad \deg \mathrm{Ker}(d_\varepsilon m) = \dim G[[t]]\, \bar 1 - \dim G[[t]]\, \bar\varepsilon =$

<u>Fact:</u> 1) $\dim \; G[[t]] \overset{T((t))}{\overset{\Psi}{t}} G[[t]] \big/ G[[t]] =$

$\left\langle \sum_{\alpha \in \Phi_+} \alpha \, , \, a \right\rangle$, where a is a dominant co-character in the W-orbit in $ch(T)^*$ corr to t.

2) Set $\delta = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$.

Then $\delta = \sum_{i=1}^{n} \lambda_i$,

where $\{\lambda_i\}$ is a basis of fundamental dominant weights.

Complete Integrability

Assume, for simplicity, that G is semi-simple and simply connected.

The fundamental weights $\lambda_1, \dots, \lambda_r$ are the heighest weight of the fund repr ρ_i. Let $\chi_i := \mathrm{tr}\,\rho_i$ be the corr trace.

$$\begin{tikzcd} T \arrow[d] & \subset & G \arrow[rr, "(\chi_1, \dots, \chi_r)"] \arrow[d] & & \mathbb{C}^r \\ T/W \arrow[rr, "\simeq"] & & G/\!/G \arrow[urr, "\simeq"'] & & \end{tikzcd}$$

Given a pair $(P, \mathcal{E}) \in M(G, c, \mathcal{O})$,

$$\begin{tikzcd} P(G) \arrow[r] & P(G)/\!/P(G) \simeq \Sigma \times (T/W) \\ \Sigma \arrow[u, "\mathcal{E}"] \arrow[ur, dashed, "\mathrm{char}(\mathcal{E})"'] & \end{tikzcd}$$

At $p \in \Sigma$, $\quad \mathrm{Ord}_p(\chi_i(\mathcal{E})) = -\lambda_i(a_p)$

dominant co-char in $\mathcal{O}_p$

Let $\underline{S_i} := \sum_{p \in \Sigma} \lambda_i(a_{\underline{p}}) \cdot p$

$$N := \bigoplus_{i=1}^{r} \mathcal{O}_\Sigma(S_i)$$

$$\deg N = \sum_p \sum_{i=1}^{r} \lambda_i(a_p) = \sum_p \frac{1}{2} \dim \mathcal{O}_p = \frac{1}{2} \dim M(G, c, \mathcal{O})$$