T. Hausel: Intersection theory on the moduli space of Higgs bundles

1. L^2 cohomology of mod sp
   - intersection #'s : math. AG/9805071
2. Cohomology ring (w/ M. Thaddeus) : math. AG/0003093 ;
   math. AG/0003094
   math. AG/0107040
3. Stringy cohomology and mirror symmetry : math. AG/0106140

Def. C - $g>1$ curve $\begin{array}{c}\Lambda\\ \downarrow\\ C\end{array}$, line bundle $\deg = 1$.

$\mathcal{N}$ = mod space of rank 2 stable bundles w/ determinant Λ

smooth proj var. $\dim = 3g-3$

Atiyah & Bott 1982 $\mathcal{N} = \mathcal{A}//\mathcal{G}$

Hitchin 1987 $\mathcal{M}$ mod space of stable Higgs bundles (E, φ)

$\det E = \Lambda$, $\varphi \in H^0(\operatorname{End} E \otimes K)$

$$\mathcal{M} = T^*\mathcal{A} ////\mathcal{G}$$

$T^*\mathcal{N} \subset \mathcal{M}$ dense open subset

$$\mathcal{M} = \operatorname{Proj}\left(\bigoplus_k H^0(T^*\mathcal{N}, (\pi^* L)^k)\right)$$

$\begin{array}{c}L\\ \downarrow\\ \mathcal{N}\end{array}$ ample line bundle over $\mathcal{N}$

$$\mathcal{M} \xrightarrow{\chi\ = \text{Hitchin map}} \operatorname{Spec}\left(H^0\left(T^*\mathcal{N}, \mathcal{O}_{T^*\mathcal{N}}\right)\right) \cong \mathbb{C}^{3g-3}$$

1) L^2-coho of $\mathcal{M}$

Problem: Determine $H^*_{L^2}(M) = \{\alpha \in \Omega_{L^2}(\mathcal{M}) \mid \Delta\alpha = 0\}$

(in H's thesis)

Thm (Hitchin 2000)

– $H^k_{L^2}(\mathcal{M}) = 0$ $\quad k \neq m = 6g-6$

– $H^m_{L^2}(\mathcal{M})$ is (anti) self dual ($*\alpha = \pm\alpha$)

$\mathrm{im}(H^m_{cpct}(\mathcal{M}) \longrightarrow H^m(\mathcal{M}))$ "$\subset$" $H^m_{L^2}(\mathcal{M})$

$\longrightarrow$ intersection form on $H^m_{cpct}(\mathcal{M})$

Thm (Hausel 1998) int form $= 0$ $\Longrightarrow$ no topological L^2 harm on $\mathcal{M}$.

Nilpotent cone $= \chi^{-1}(0) = [N] \cup$ .. other components $[N_i]$

~~has singularity over components~~

$\longrightarrow$ give basis of $H^k_{cpct}(\mathcal{M})$

Formula: $\#(N_i \cap N_j) = \chi(N_i \cap N_j)\,(-1)^{\dim(N_i \cap N_j)}$

if $N_i, N_j, N_i \cap N_j$ smooth.

Pf: Similar to k-stable bundles.

Virtual Dirac bundle $\mathbb{D}_k = \pi_!(\mathbb{E}\otimes\Lambda^{k-1} \longrightarrow \mathbb{E}\otimes\Lambda^{k-1}\otimes K) \in \mathcal{K}(M)$

consider objects like

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\Phi"] \arrow[d] & \mathbb{E}\otimes K \\ \mathcal{M} & \end{tikzcd}$$

"$\underset{\overset{\shortparallel}{0}}{c_{3g-3}}(\mathbb{D}_k) = [N_k]$"

2) Cohomology ring (w/ M. Taddeus)

$H^*(N)$

Thm (H–T, 2000) $\quad \Gamma = \mathbb{Z}_2^{2g} = \mathrm{Jac}_2(C) \curvearrowright \mathcal{M}$

$$H^*(\mathcal{M})^\Gamma = \mathbb{Q}[\alpha, \psi_i, \beta] / I$$

$\deg \alpha = 2 \quad \deg \psi_i = 3 \quad \deg \beta = 4$

I is described explicitely.

Idea of the generation of $H^*(\mathcal{M})$

Consider oid variety

$$\mathcal{M}_\infty = \bigcup_k \mathcal{M}_k = \{ (E, \varphi) \quad \exists p \in C \quad \varphi \in \mathrm{End}_0 E \otimes K \otimes \mathcal{O}(kp) \text{ (pole of degree } k) \}$$

(~~Donagy – Markman~~ Bottacin 1994)

$\mathcal{M}_0 \subset \mathcal{M}_\infty$ is a symplectic leaf.

1) $\mathcal{M}_\infty \simeq B\bar{\mathcal{G}}$ (gauge grp.)

2) $H^*(\mathcal{M}_\infty) \longrightarrow\!\!\!\!\!\rightarrow H^*(\mathcal{M})$ (breaks down for rank ≥ 2)

Remarks about I : $\mathcal{M}_k$ is used

3) Stringy cohomology, ~~[illegible]~~ mirror symmetry

$$J_2(C) = \Gamma = \mathbb{Z}_2^{2g} \circlearrowright \mathcal{M}$$

$\mathcal{M}/\Gamma = \hat{\mathcal{M}}$ - mirror image of $\mathcal{M}$

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "\chi"] & & \mathcal{M}/\Gamma \arrow[dl, "\hat{\chi}"'] \\ & \mathbb{C}^{3g-3} & \end{tikzcd}$$

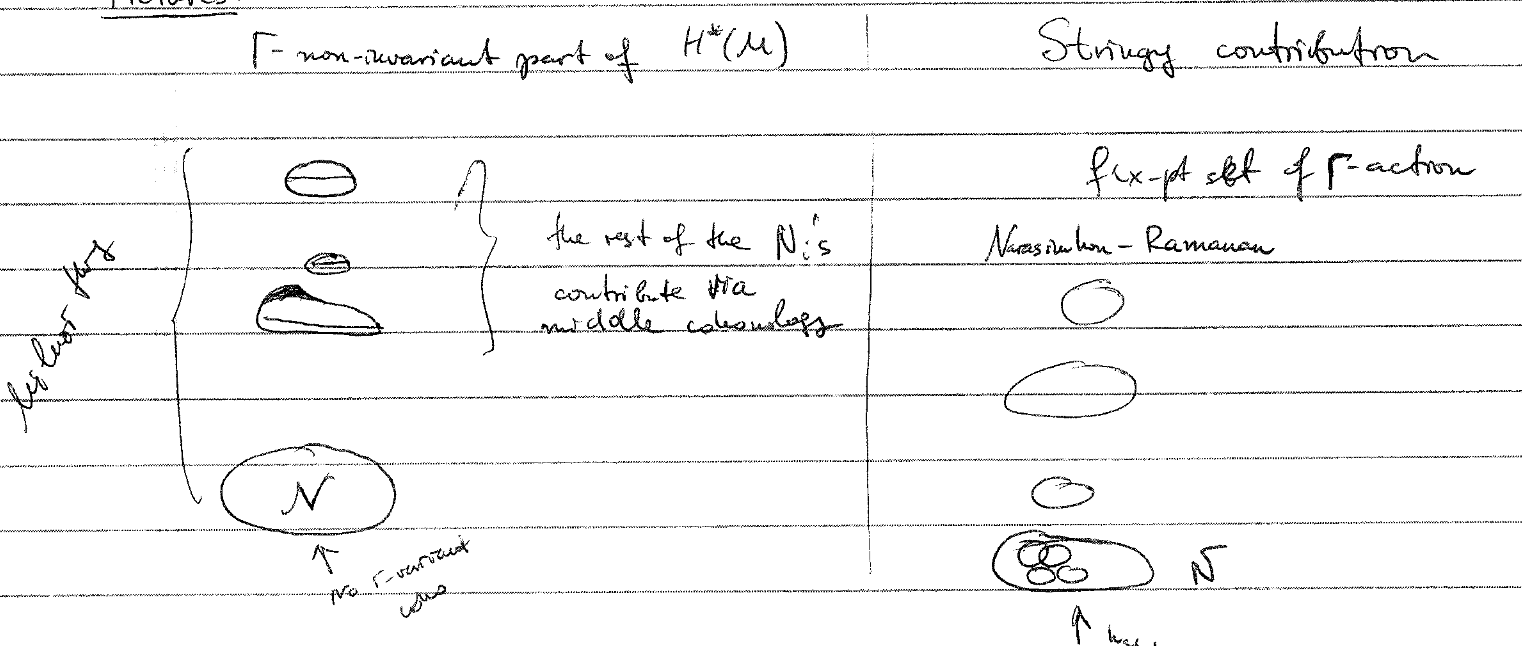
" $\chi^{-1}(p) = (\hat{\chi}^{-1}(p))^*$ "

$\Rightarrow$ Strominger - Yau - Zaslow for mirror symmetry.

Thm (H-T, 2001) $h^{p,q}(\mathcal{M}) = h^{p,q}_{stringy}(\hat{\mathcal{M}}, B)$

natural determinantal B-field.

Pictures:



Calculation for $r=3$.

Framework for all 3 aspects

Question: Is there a natural framework which will include 3 aspects of the intersection theory of $\mathcal{M}$.

1) $\overline{\mathcal{M}}$ - compactification (symplectic)
$H^*(\overline{\mathcal{M}})$ sees $H^*(\mathcal{M})$ and the intersection form on $H^{m}_{cpt}(\mathcal{M})$

$$\mathcal{M} \subset \mathbb{P}^k \times \mathbb{C}^{3g-3} \subset \mathbb{P}^k \times \mathbb{P}^{3g-3}$$

2) derived category of coherent sheaves on $\mathcal{M}$

3) Topological QFT s.t. for a Riemann surface
$H^*_{S^1}(\mathcal{M})$ equivariant
or $K^*_{S^1}(\mathcal{M})$