Variations of Pure Hodge Structure (Families of smooth proj varieties

Higgs Bundles
Yang-Mills Eq

Simpson's Correspondence

Kodaira-Spencer-Map

acts via Higgs Field θ

Harmonic metrics

Hodge Metric

VHS

$\mathbb{C}$VHS, $\mathbb{C}^*$-fixed points

$(E, \bar\partial+\theta) \to (E, \bar\partial+\lambda\theta)$

Variations of Mixed Hodge Structure

Filtered Higgs Bundles
String Theory
Quantum Cohomology

Here, we have the action have but not θ; do these? Kodaira-Spencer Map

Deformation Theory

Analogs of Simpson's Correspondence

Filtered Harmonic Metrics

mixed Hodge metric

VGpMHS

$\mathbb{C}$VMHS $\mathbb{C}^*$-fixed points

$(E, \bar\partial+\theta) \to (E, \bar\partial+\lambda\theta)$

Motivation

Original Problem, which lead to above.

Part 1. Schmid's Nilpotent orbit Theorem

Outline $f: \mathfrak{X} \to S$ complex analytic family of compact Kähler manifolds

$\rightsquigarrow$ $\mathcal{H} = R^k_{f*}(\mathbb{Q}) \otimes \mathcal{O}_S$ is a VHS of weight k

Hodge-Decomp Theorem; Cf. Ramadas's Talk Lecture 1

(i) Each fiber $\mathcal{H}_s = H^k(\mathfrak{X}_s, \mathbb{C})$ is a pure, polarized Hodge struct of weight k.

(ii) The resulting Hodge filtration $\mathcal{F}^p = \bigoplus_{r \geq p} \mathcal{H}^{r,k-r}$ is horizontal, with respect to the ~~underlying~~ connection ∇ of the underlying local system $\mathcal{H}_{\mathbb{Q}} = R^k_{f*}(\mathbb{Q})$.

Data for VHS

By parallel translation to a fixed reference fiber $H = \mathcal{H}_{s_0}$, one then obtains data

(i) Monodromy rep $\rho: \pi_1(S, s_0) \to$ [redacted] $\mathrm{Aut}_{\mathbb{Q}}(Q) \to H$ (bilinear form) $= \mathcal{H}_{s_0}$.

Image $(\rho) = \Gamma$.

(ii) Period map $\varphi: S \to \mathcal{D}/\Gamma$, which is holomorphic, horizontal, locally liftable. ($\mathcal{D}$ ← classifying space of polarized Hodge struct)

Remark: $\mathcal{D}$ is a submanifold of a suitable flag variety, therefore holomorphic, horizontal mean that for any local lift F of φ to $\mathcal{D}$,

$\frac{\partial}{\partial \bar{s}_j} F^p \subset F^p$, $\frac{\partial}{\partial s_j} F^p \subset F^{p-1}$.

Specializing to $S = \Delta^*$, one then has the following (local) description of the asymptotic behavior of a VHS:

(a) The Hodge metric $h_F(u,v) = G(C_F u, \bar{v})$, $C_F|_{H^{p,q}} = i^{p-q}$ defined by $F \in \mathcal{D}$ on the ambient vector space $H = H_{\mathbb{C}}$ induces a metric h (Hodge metric) on $\mathcal{D}$

(b) In this case, we get a lift

$$\begin{tikzcd} \text{coord} = z & U \arrow[r, "F(z)"] \arrow[d, "s = e^{2\pi i z}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[d] & U = \text{upper-half plane} \\ \text{coord} = s & \Delta^* \arrow[r, "\varphi(s)"] & \mathcal{D}/\Gamma & \end{tikzcd}$$

which satisfies the quasi-periodicity condition

$$F(z+1) = e^N \cdot F(z)$$

(N = monodromy log), and hence

$$\psi(z) = e^{-zN} \cdot F(z)$$

is periodic ($\psi(z+1) = \psi(z)$) and therefore descends to a map $\psi(s) : \Delta^* \to \check{\mathcal{D}}$ the compact dual of $\mathcal{D}$

(c) (1st part of Schmid's Nilpotent Orbit Theorem) asserts that

$$F_\infty = \lim_{s \to 0} \psi(s)$$

exists in $\check{\mathcal{D}}$.

(d) (2nd part of Schmid's nilpotent orbit theorem) $e^{zN} \cdot F_\infty \in \mathcal{D}$ for $\mathrm{Im}(z) \gg 0$.

(e) (3rd part nilpotent orbit theorem)

$$d_{\mathcal{D}}(F(z), e^{zN} \cdot F_\infty) < K\, \mathrm{Im}(z)^{\beta}\, e^{-2\pi \mathrm{Im}(z)}$$

analogs describe singularities of the period map in terms of monodromy. Question ① what does commuting part of period map describe? ② What is the analog of

Part 2: Goal: Find analogs of this result for the VGPMHS, which is what you get if you replace family of smooth varieties by family of singular (or quasi-projective) varieties. ∃ DMJ, vol 110, 2001. Hodge module

Recall: MHS (F, W) consists of an increasing filtration

$$0 \subseteq \cdots \subseteq W_k \subseteq W_{k+1} \subseteq \cdots \subseteq V$$

of a finite dimensional $\mathbb{Q}$-vector space $V = V_{\mathbb{Q}} \otimes \mathbb{C}$ and a decreasing filtration

$$0 \subseteq \cdots \subseteq F^p \subseteq F^{p-1} \subseteq \cdots \subseteq V$$

such that F induces a pure Hodge structure of weight k on $Gr^W_k = W_k/W_{k-1}$ via the rule

$$F^p Gr^W_k = \frac{F^p \cap W_k + W_{k-1}}{W_{k-1}}$$

Note: Def of a VGPMHS formally analogous with W + Griffiths transversality flat.
Definition next section using classifying space.

Part 3: Classifying Space

As in the pure case, a VGPMHS $\mathcal{V} \to S$ is equivalent to data

(a) monodromy rep ρ, $Im(\rho) = \Gamma$

(b) period map $\varphi : S \to M/\Gamma$ from S into the quotient of a classifying space of graded-polarized MHS, which is holomorphic, horizontal and locally liftable.

Definition of M: Data $V = V_{\mathbb{Q}} \otimes \mathbb{C}$, W = increasing filtration of V, $\{Q_k\}$ = collection of non-deg bilinear forms on Gr^W_k of alt parity $(-1)^k$, $\dim W_k = \sum_{r+s \le k} h^{r,s}$, $h^{r,s} \ge 0$, $h^{r,s} = h^{s,r}$;

~~[illegible]~~

$M = \{ F$ dec filt of $V \mid (F, W)$ is a MHS graded-polarized by $\{Q_k\}$ such that $\dim F^p Gr^W_k = \sum_{r \ge p} h^{r,k-r} \}$

~~Proof~~

Theorem: Let $GL(V)^W = \{ g \in GL(V) \mid g : W_k \to W_k \ \forall k \}$. Then

$$G = \{ g \in GL(V)^W \mid Gr(g) \in Aut_{\mathbb{R}}(Q_k) \}$$

~~acts transitively on M.~~

Theorem: M is a complex manifold upon which G acts transitively by biholomorphism.

Proof $\check{\mathcal{F}}$ = all flags such that $\dim_{\mathbb{C}} F^p = \sum_{r \geq p} h^{r,s}$

$\cup$

$\check{\mathcal{F}}(\varphi)$ = all flags $F \in \check{\mathcal{F}}$ such that $\dim_{\mathbb{C}} F^p_{G} = \sum_{r \geq p} h^{r,s}$

$\cup$

compact dual of m not compact $\longrightarrow$ $\check{m}$ = $F \in \check{\mathcal{F}}(\varphi)$ such that $Q_k(F^p \ldots, F^{k-p+1} \ldots) = 0$

$\cup$

m

$GL(V)$ acts trans on $\check{\mathcal{F}}$

$GL(V)^{\varphi}$ acts trans on $\check{\mathcal{F}}(\varphi)$

$G_{\mathbb{C}}$ acts trans on $\check{m}$, $G_{\mathbb{C}} = \{ g \in GL(V)^{\varphi} \mid \ldots \in \operatorname{Aut}_{\mathbb{R}}(Q) \}$

G acts trans on m.

(Comment about $m_{\mathbb{R}}$)

Campbell-Baker-Hausdorff Formula

$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y] + O^3}$ i.e. Error is $\|E\| \leq c(\|X\|+\|Y\|)^3$

Deligne – Hodge Bundles

Pure Case: Basic C^∞-infinity bundles over $\mathcal{D}$ are

$$\mathcal{H}^{p,q}_F = H^{p,q}_F \subset V = \text{ambient vector space}$$

Note: $\mathcal{H}^{p,q} \cong \mathcal{F}^p/\mathcal{F}^{p+1}$ and therefore inherits a complex structure, since $\nabla^{0,1}(\mathcal{F}^p) \subseteq \mathcal{F}^p \otimes \mathcal{E}^{0,1}$. However $\mathcal{H}^{p,q}$ is not holomorphic rel $\nabla^{0,1}$ acting on $V \times \mathcal{D} \to \mathcal{D}$

Mixed Case: Analogous result.

Theorem (Deligne): Let (F, W) be a mixed Hodge structure on $V = V_{\mathbb{R}} \otimes \mathbb{C}$. Then, there exists a unique, functorial bigrading

$$V = \bigoplus I^{p,q}, \quad I^{p,q} = I^{p,q}_{(F,W)}$$

such that

1. $F^p = \bigoplus_{a \geq p} I^{a,b}$
2. $W_k = \bigoplus_{a+b \leq k} I^{a,b}$
3. $\overline{I^{p,q}} = I^{q,p} \mod \bigoplus_{r<q,\, s<p} I^{r,s}$.

Applying this construction pointwise to $\mathcal{M}$, to obtain Deligne – Hodge Bundles

$$\mathcal{I}^{p,q} \to \mathcal{M}, \quad \mathcal{I}^{p,q}_F = I^{p,q}_{(F,W)}.$$

Goal: Understand first order behavior of $I^{p,q} \to \mathcal{M}$ relative to ∇.

Observations

① $g \in G_{\mathbb{R}} = G \cap GL(V_{\mathbb{R}}) \Rightarrow I^{p,q}_{(g.F,W)} = g.I^{p,q}_{(F,W)}$

② In contrast to pure case, $G_{\mathbb{R}}$ does not act transitively on $\mathcal{M}$ (in general).

Lemma: By functoriality, $F \in \mathcal{M}$ induces a MHS on

$$\mathfrak{g}_{\mathbb{C}} = Lie(G_{\mathbb{C}}) = Lie(G_{\mathbb{R}}) \otimes \mathbb{C}.$$

and hence a decomposition ← By $\mathfrak{g}^{r,s} = I^{r,s}(\mathfrak{g}_{\mathbb{C}})$

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r,s} \mathfrak{g}^{r,s}$$

(with $r+s \leq 0$ since $\mathfrak{g}^{r,s}(W_k) \subseteq W_{k+r+s}$)

Moreover, if

$$\Lambda^{-1,-1}_{(F,W)} = \bigoplus_{r,s<0} \mathfrak{g}^{r,s}$$

then

$$\lambda \in \Lambda^{-1,-1}_{(F,W)} \Rightarrow I^{p,q}_{(e^{\lambda}.F,W)} = e^{\lambda}.I^{p,q}_{(F,W)}$$

Proof (Sketch):

① $\overline{\Lambda^{-1,-1}} = \Lambda^{-1,-1}$

② Set $J^{p,q} = e^{\lambda}.I^{p,q}_{(F,W)}$ and then show that $\overline{J^{p,q}} = J^{q,p} \bmod \bigoplus_{r<q,\, s<p} J^{r,s}$

[Kaplan]

Theorem: Given $F \in \mathcal{M}$ and $g \in G$ there exists a distinguished, C^∞ decomp

$$g = g_{\mathbb{R}} e^{\lambda} e^{\varphi}$$

with $g_{\mathbb{R}} \in G_{\mathbb{R}}$, $\lambda \in \Lambda^{-1,-1}_{(F,w)}$, $\varphi \in \mathrm{Lie}(G^F)$

Corollary: For any $F \in \mathcal{M}$, $\mathcal{M} = G_{\mathbb{R}} \exp(\Lambda^{-1,-1}_{(F,w)}).F$.

In particular,

$$I^{p,q}_{(g.F,w)} = I^{p,q}_{(g_{\mathbb{R}} e^{\lambda}.F,\, w)}$$

$$= g_{\mathbb{R}} e^{\lambda} I^{p,q}_{(F,w)}$$

$$= g\, e^{-\varphi} I^{p,q}_{(F,w)}.$$

Theorem [Pearlstein]: Let F be an element of $\mathcal{M}$. Then, there exists a natural $\mathbb{R}$-vector space $\Phi_F \subset \mathrm{Lie}(G_{\mathbb{C}}^F)$ such that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \oplus \sqrt{-1}\left(\Lambda^{-1,-1}_{(F,w)} \cap \mathfrak{g}_{\mathbb{R}}\right) \oplus \Phi_F^{\mathfrak{g}}$$

Def of Φ_F: $\begin{cases} \eta_+ = \bigoplus_{r\geq 0,\, s<0} \mathfrak{g}^{r,s}, & \eta_- = \bigoplus_{s\geq 0,\, r<0} \mathfrak{g}^{r,s} \\ \eta_0 = \mathfrak{g}^{0,0}, & \Lambda^{-1,-1} = \bigoplus_{r<0,\, s<0} \mathfrak{g}^{r,s} \end{cases}$

~~Then~~ $\mathfrak{g}_{\mathbb{C}} = \eta_+ \oplus \eta_0 \oplus \eta_- \oplus \Lambda^{-1,-1}$

$$\Phi_F = \eta_+ \oplus \{ x \in \eta_0 \mid \pi_0(\bar{x}) = -\pi_0(x) \}$$

$\pi_+, \pi_0, \pi_-, \pi_\Lambda$ = projection operators

Corollary: $\exists$ neigh U of zero in $\mathfrak{g}_G$ such that
$u \in U \Rightarrow e^u = g_{\mathbb{R}}(u)\, e^{\lambda(u)}\, e^{\phi(u)}$
with $g_{\mathbb{R}}(u) \in G_{\mathbb{R}}$, $\lambda(u) \in \sqrt{-1}\,(\mathfrak{n}^{1,-1} \cap \mathrm{Lie}\, G_{\mathbb{R}})$, $\phi(u) \in \Phi_F$.

Next: $\mathfrak{g}_G = \mathrm{Lie}(G^F) \oplus \mathfrak{q}_F$
$\hookrightarrow \mathfrak{q}_F = \mathfrak{n}_- \oplus \mathfrak{n}^{-1,1}$

$\Rightarrow$ $\exists$ neigh U_F of zero in $\mathfrak{q}_F$ such that
$u \in U_F \to e^u.F \in \check{M}$ is biholomorphism.
(Because $G_{\mathbb{C}}$ acts trans on $\check{M}$ and M is open in $\check{M}$)

2nd order in $u, \bar{u}$

Lemma: $u \in \mathfrak{q}_F \Rightarrow \phi(u) = -\pi_+(\bar{u}) + O^2(u)$
Sketch of Proof:

$\mathfrak{g}_G = \mathfrak{g}_{\mathbb{R}} \oplus \sqrt{-1}\,(\mathfrak{n}^{-1,-1} \cap \mathfrak{g}_{\mathbb{R}}) \oplus \Phi_F$ (*)
$\therefore$ up to first order, the corresponding decomp
$e^u = g_{\mathbb{R}}(u)\, e^{\lambda(u)}\, e^{\phi(u)}$
is defined by proj $\mathfrak{q}_F \to \Phi_F$ rel (*).

<u>Theorem</u>: Let $\mathcal{V} \to S$ be a VGPMHS, and

$$\mathcal{I}^{p,q} = \mathcal{I}^{p,q}_{(F,W)}$$

be the associated system of C^∞-subbundles.

Then, ~~[illegible]~~

$$\nabla : \mathcal{E}^0(\mathcal{I}^{p,q}) \to \mathcal{E}^{0,1}(\mathcal{I}^{p+1,q-1}) \oplus \mathcal{E}^{0,1}\Big(\bigoplus_{b \le q} \mathcal{I}^{p,b}\Big) \oplus \mathcal{E}^{1,0}(\mathcal{I}^{p,q})$$
$$\oplus\, \mathcal{E}^{1,0}\Big(\bigoplus_{b \le q} \mathcal{I}^{p,b}\Big)$$

where ∇ is the Gauss-Manin connection of $\mathcal{V}$.

(C VHS unpolarized)

<u>Corollary</u>: The sub-bundles $\mathcal{U}^p = \bigoplus_q \mathcal{I}^{p,q}$ form a $\mathbb{C}$VHS (unpolarized) rel to ∇.

$(\Theta)^2 = 0$

<u>Corollary</u>: The underlying C^∞-bundle of $\mathcal{V}$ carries a natural Higgs bundle structure $(\bar\partial + \Theta)$ which is a fixed point of the $\mathbb{C}^*$ action $(\bar\partial + \Theta) \to (\bar\partial + \lambda\Theta)$.

$$\nabla = \tau + \bar\partial + \partial + \Theta$$

<u>Corollary</u>: Mixed case, $\tau = (\bar\Theta)^{\mathcal{V}}$.

§5: Have Higgs field on ~~[illegible]~~ interior $\mathcal{V} \to S$ }
now want Higgs field at ∞ }

§6: Admissible VGPMHS. (original problem)

Recall: Given a flat vector bundle $E \to \Delta^{*N}$ with unipotent monodromy, $\exists$ canonical extension $E^c \to \Delta^N$ relative to which ∇ has at worst simple poles with nilpotent residues along

$$D = \Delta^N - \Delta^{*N}$$

Explicit: Select coord rel $D = \{ s_1 \cdots s_N = 0\}$ then E^c is the bundle generated by the sections

$$\sigma^c = e^{\sum_j \frac{1}{2\pi i}(\log s_j) N_j} \sigma, \quad \sigma - \text{flat, multivalued section of } E.$$

$$\Rightarrow \nabla^c = \nabla - \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{ds_j}{s_j} \otimes N_j$$

is a flat connection on E^c.

Def: A VGPMHS $\mathcal{V} \to \Delta^*$ with unipotent monodromy is admissible iff

① F extends to $\mathcal{V}^c$

② The relative weight filtration ${}^r W = {}^r W(N, W)$ exists

Theorem (Deligne): ①+② $\Rightarrow$ $(F_\infty, {}^r W)$ define a MHS relative to which N is a morphism of type $(-1,-1)$.

Def: $\mathcal{V} \to \Delta^{*N}$ is admissible iff the restriction of $\mathcal{V}$ to every meromorphically embedded disk is admissible

Remark: I don't like this def.

Theorem [Kaplan-Pearlstein]: A unipotent VGPMHS $\mathcal{V} \to \Delta^{*}$ is admissible iff the grading $\mathcal{Y}$ of $\mathcal{V}$ which acts as multiplication by $(p+q)$ on $I^{p,q}$ extends continuously to $\mathcal{V}^c$.

Anyway: Given an admissible VGPMHS $\mathcal{V} \to \Delta^{*N}$ as above + coord choice, parallel translation under ∇^c to the central fiber V_1 of $\mathcal{V}^c$ one obtains

① Rational struct $V_{\mathbb{Q}}$

② Rational weight filt W

③ Rational, non-deg bilinear forms $\mathcal{Q}_k$ on Gr^W_k

④ Untwisted period map $\varphi: \Delta^{*N} \to \mathcal{M}$, which extends holomorphically to $\Delta^N \to \mathcal{M}$.

such that

⑤ ${}^rW = {}^rW(N, W)$ is constant along $\mathcal{C} = \{\sum a_j N_j \mid a_j > 0\}$

⑥ $(F_\infty, {}^rW)$ is a MHS rel which each N_j is a $(-1,-1)$ morphism.

In particular

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_a \mathfrak{p}_a, \qquad \mathfrak{p}_a : U^p_\infty \to U^{p+a}_\infty$$

where

$$U^q_\infty = \bigoplus_q I^{p,q}_{(F_\infty, N)}$$

Moreover, since $F^{p+1}_\infty \oplus U^p_\infty = F^p_\infty$,

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G^{F_\infty}_{\mathbb{C}}) \oplus \mathfrak{q}_\infty, \qquad \mathfrak{q}_\infty = \bigoplus_{a<0} \mathfrak{p}_a.$$

Therefore, since $G_{\mathbb{C}}$ acts transitively on $\check{M}$, we can (locally) write (unique)

$$\psi(s_1, \dots, s_n) = e^{\Gamma(s_1, \dots, s_n)} . F_\infty$$

for $\Gamma(s_1, \dots, s_n)$ with values in $\mathfrak{q}_\infty$ and $\Gamma(0) = 0$.

$$\Rightarrow \quad F(z_1, \dots, z_n) = e^{\sum_j z_j N_j} e^{\Gamma(s_1, \dots, s_n)} . F_\infty$$

Note: $N_j \in \mathfrak{p}_{-1} \Rightarrow e^{\sum_j z_j N_j} e^{\Gamma(s_1, \dots, s_n)} = e^{X(z_1, \dots, z_n)}$

$X(z_1, \dots, z_n) \in \mathfrak{q}_\infty$.

$\therefore$ Horizontality $\Rightarrow$ $e^{-X} \partial e^X \in \mathfrak{p}_{-1}$

$$\Rightarrow \quad e^{-X} \partial e^X = \underbrace{\Omega + \partial \Gamma_{-1}}_{\partial X_{-1}}, \qquad \Omega = \frac{1}{2\pi i} \sum_j \frac{ds_j}{s_j} \otimes N_j$$

Gromov – Witten Potential

holomorphic in $(q_1, \ldots, q_N)$ on Δ^N.

$$\Phi(u_1, \ldots, u_N) = \frac{1}{6}\int_X \omega^3 + \Phi_{hol}(q_1, \ldots, q_N)$$

$$\omega(u_1, \ldots, u_N) = \sum u_j T_j$$

$$q_j = e^{2\pi i u_j}$$

Comes from Quantum Cohomology.

Gromov–Witten invariants

$$\Phi_{hol}(q_1, \ldots, q_N) = \frac{1}{(2\pi i)^3} \sum_{\beta \in H^2(X,\mathbb{Z})\setminus\{0\}} \langle I_{0,0,\beta}\rangle e^{2\pi i \int_\beta \omega}$$

A-model variation:

① Dubrovin connection on $V = H \times \overline{D}_\sigma$, $H = \bigoplus_p H^{p,p}(X)$

$\nabla = d + A$, $A_{\frac{\partial}{\partial u_j}} \alpha = \underbrace{T_j * \alpha}_{\text{small quantum product.}}$

② Hodge filtration

$$F^p = \bigoplus_{a \le 3-p} H^{a,a}(X, \mathbb{C})$$

③ Polarization: Pair $H^{p,p}(X)$ and $H^{3-p,3-p}(X)$ by

$$Q(\alpha, \beta) = (-1)^p \int_X \alpha \wedge \beta$$

④ Integral Structure: $\nabla^c = \nabla - \frac{1}{2\pi i}\sum_{j=1}^N \frac{dq_j}{q_j} \otimes N_j$

$N_j = -$ (cup prod with T_j)

$$\sigma = e^{-\sum \frac{1}{2\pi i}\log(q_j) N_j}\, \sigma^c$$

integral if

ⓐ σ^c is ∇^c flat

ⓑ $\sigma^c(0) \in H(X, \mathbb{Z})$.

<u>Theorem</u>: For the Function Γ (or Higgs Field $\partial \mathcal{I}_1$) associated to the A-model variation,

$$\Gamma_3(T_0) = -2\,\mathcal{I}_{n,1}\,T_0^\vee$$

where $T_0 = 1 \in H^0(X, \mathbb{C})$ and $T_0^\vee$ is the Poincaré Dual of T_0

Hodge filt — Monodromy weight filt.

<u>Remark</u>: There is a similar result for the B-model: Namely (by Deligne's observation) (F, W) defines a VGPMHS of Hodge-Tate type. If θ is the associated Higgs Field, then (after translating to central-fiber) $\theta = \partial \mathcal{I}_1 \mod \bigoplus_{a \geq 2} p_a$.

Part 7: A-model Variation (Brief)

X = smooth Calabi-Yau threefold

Kähler Cone $K(X) = \{ \omega \in H^2(X, \mathbb{R}) \mid \omega \text{ is Kähler} \}$

Complexified Kähler Space $K_{\mathbb{C}}(X) = \dfrac{\{\omega \in H^2(X,\mathbb{C}) \mid \mathrm{Im}(\omega) \in K(X)\}}{H^2(X,\mathbb{Z})}$ (= torsion free part)

Large Radius Limit Point (of $K_{\mathbb{C}}(X)$): Simplicial Cone $\sigma \subseteq H^2(X, \mathbb{R})$ which is of maximal dimension, and satisfies $\mathrm{Int}(\sigma) \subseteq K(X)$.

Compactification of $K_{\mathbb{C}}(X)$ near σ (= large radius limit point).
Pick a basis $T_1, \ldots, T_N$ of $(H^2(X,\mathbb{Z})/\text{torsion}) \cap K(X)$ which generate σ.

$\Rightarrow \mathrm{Int}(\sigma) = \{ \sum_j a_j T_j \mid a_j > 0 \}$

$\Rightarrow \mathcal{U}^{\sim}_{\sigma} = \{ \sum_j u_j T_j \mid \mathrm{Im}(u_j) > 0 \}$
projects onto a neighborhood D_σ of $K_{\mathbb{C}}(X)$
via $u \in \mathcal{U}^{\sim}_{\sigma} \to [u] \in K_{\mathbb{C}}(X)$

$\Rightarrow$ get completion $D_\sigma \hookrightarrow \overline{D_\sigma}$
via $\Delta^{*N} \hookrightarrow \Delta^N$ defined by
standard covering map $(u_1, \ldots, u_N) \in \mathcal{U}^{\sim}_{\sigma} \to$
$(e^{2\pi i u_1}, \ldots, e^{2\pi i u_N}) \in \Delta^{*N}$

Def: A VMHS is unipotent if the reduced variations on Gr^W are constant.

Corollary: If $\mathcal{V}$ is unipotent then $\nabla^{0,1} = \bar{\partial}$ and hence each $\mathcal{V}^p$ is holomorphic relative to $\nabla^{0,1}$: Proof $\tau = (\bar{\Theta})^{q}$ and hence comes from action on Gr^W, which is trivial.

Corollary: A unipotent VMHS $\mathcal{V} \to \bar{X}$ is determined by the following data

1. ∇ (only, as equiv to monodromy rep)
2. Ref fiber $\mathcal{V}_x$ (algebraic).
3. Θ.

Proof: The only thing which isn't flat wrt to ∇ (and hence recoverable from ∇ and $\mathcal{V}_x$) is F^p. But, F^p is parallel under $\bar{\partial} + \partial$ and

$$R_{\bar{\partial}+\partial} = -(\tau \wedge \Theta + \Theta \wedge \tau)$$

$\therefore$ $\tau = 0 \Rightarrow R_{\bar{\partial}+\partial} = 0 \Rightarrow$ recover F^p by parallel translation under

$$\bar{\partial} + \partial = \nabla - \Theta. \quad (\text{again } \tau = 0)$$