

M. Thaddeus: Higgs bundles spaces as mirror partners 3/30/2002
 w/ T. Hausel

$$M, \hat{M}$$

$$h^{p,q}(M) = h^{n-p,q}(\hat{M})$$

Defn. $M \xleftarrow{\text{SYZ}} \hat{M}$

$$\begin{array}{ccc} M & & \hat{M} \\ & \searrow \mu & \swarrow \hat{\mu} \\ & N & \end{array}$$

$$\dim_{\mathbb{R}} N = n$$

s.t. for reg. value x of μ and $\hat{\mu}$,

$$L_x = \mu^{-1}(x) \quad \text{and} \quad \hat{L}_x = \hat{\mu}^{-1}(x)$$

are special Lagrangian tori

$$L_x \simeq (S^1)^n$$

$$\omega|_{L_x} = 0$$

$$L_x \simeq \text{Hom}(\pi_1(\hat{L}_x), \mathbb{C}^*)$$

$$\hat{L}_x = \text{Hom}(\pi_1(L_x), \mathbb{C}^*)$$

$$\text{Im } \Omega|_{L_x} = 0$$

depend smoothly on x .

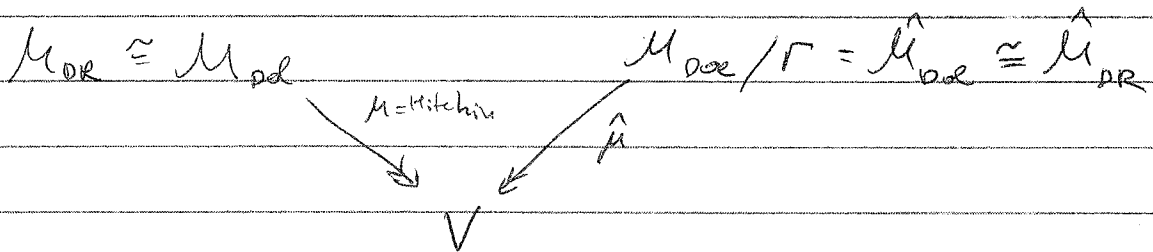
Lemma M -hyperkähler, LCM complex Lagrangian
w.r.t $J \Rightarrow$ special Lagrangian w.r.t. J_2

Let C be a smooth curve genus $=g$

$$M_{DR} = \{ \text{irreducible local systems } (SL_r) \}$$

$$M_{\text{rel}} = \{ \text{stable } SL_r\text{-Higgs bundles} \}$$

$$= \{ (E, \varphi) \mid \Lambda^r E \cong \mathcal{O}, \text{tr } \varphi = 0, \text{stable} \}$$



$$\mu^{-1}(x) = \text{Prym} = N_m^{-1}(0)$$

$$\uparrow \text{norm map} : N_m : \text{Pic}^0(\tilde{C}) \rightarrow \text{Pic}^0(C)$$

$$\hat{\mu}^{-1}(x) = ?$$

$$0 \rightarrow \text{Prym} \rightarrow \text{Pic}^0(\tilde{C}) \xrightarrow{N_m} \text{Pic}^0(C) \rightarrow 0$$

$$0 \rightarrow \text{Pic}^0 C \xrightarrow{\pi^*} \text{Pic}^0(\tilde{C}) \rightarrow \text{Prym}/\Gamma \rightarrow 0$$

$$\Gamma = \text{Prym} \cap \pi^* \text{Pic}^0 C = \text{Pic}^0 C[\mathbb{F}] \Rightarrow M_{\text{rel}}/\Gamma$$

$$\hat{M}_{DR} = M_{DR}/\Gamma = \{ \text{mod } PGL_r \text{ local systems} \}$$

$$\hat{M}_{Doe} = M_{Doe} / \Gamma = \{ \text{stable } PGL_r \text{ Higgs bundles} \}$$

(topological trivial)

$SL_r \leftrightarrow PGL_r$ are dual (Langlands)

Proposition

$$M_{DR} \xleftrightarrow{SYZ} M_{DR}^1$$

for any reductive group G and the dual group \hat{G}

- (What is the fiber of μ :
- $(SL_r) \Rightarrow \text{Prym}$
 - $(G) \Rightarrow \text{Hom}_W(\Lambda, \text{Pre}^{\circ} \mathbb{C})$
"central cover"
 - $(\hat{G}) \Rightarrow \text{Hom}_W(\tilde{\Lambda}, \text{Pre}^{\circ} \mathbb{C}) !$

Extend the def of mirror symmetry :

replace

$$M \leftrightarrow (M, B)$$

$$\hat{M} \leftrightarrow (\hat{M}, \hat{B})$$

where B & \hat{B} are B -fields

Def of (Gerbe) : A -abelian grp

An (flat) A -gerbe B on X :

is a sheaf of categories locally isomorphic to the category of bundles w/ structure group A
(flat)

(this simplified version is enough for our purposes)

E.g. $\Psi = \text{PGL}$ -bundle on X

$B =$ category of liftings of Ψ to SL_r bundle

Any two liftings differ by tensorisation by a \mathbb{Z}_r bundle

$\Rightarrow B$ is \mathbb{Z}_r gerbe.

Reason:

$$0 \rightarrow \mathbb{Z}_r \rightarrow \text{SL}_r \rightarrow \text{PGL}_r \rightarrow 1$$

Then (Giraud) " "

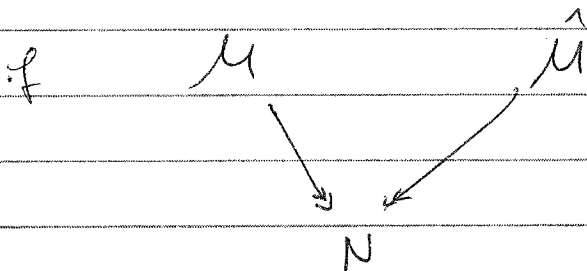
$$\left\{ \begin{array}{l} \text{Isom classes of} \\ \text{flat } A\text{-gerbes} \\ \text{on } X \end{array} \right\} = H^2(X, A)$$

(B-fields) \leftrightarrow (flat Gerbes)

- An automorphism is an aut of sheaf of categories.
- Any aut. is a tensorisation by an A -bundle.
- 2 aut.'s are equivalent if the bundles are \cong .
- Equiv classes of aut.'s = $H^1(X, A)$
- Trivialization is an iso to the trivial bundle.
- Two triv's are equivalent if $f \circ g^{-1} \cong \text{id}$.

Define $\text{Triv}(X, B) = \{ \text{equiv. cl. of triv's} \} =$
 $=$ a torsor for $H^1(X, A)$
 (over a point)

Def'n: $(M, B) \xrightarrow{\text{SYZ}} (\hat{M}, \hat{B})$



M -Elohi-Yau

$B =$ flat $U(1)$ gerbe on M

everything else is the same except:

- $L_x = \text{Triv}(\hat{L}_x, \hat{B})$

- $\hat{L}_x = \text{Triv}(L_x, B)$

(N. Hitchin: "lectures on special Lagrangians on manifolds")

- depends smoothly on x

Define $M_{\text{Dol}}^d = \{ \text{stable } (E, \varphi) \mid \wedge^r E \cong \mathcal{O}(dc), \text{tr } \varphi = 0 \}$
 $c =$ base pt of C

$M_{\text{Dol}}^g = \{ \text{mod. local systems on } C \{c\} \dots \}$

Γ^g acts (finite ab. grp.) $\hat{M} = M / \Gamma$ (PR, Dol)

Models of PGL_r Higgs = $\bigsqcup_{d=0}^r \hat{\mathcal{M}}_{\mathbb{P}^1}^d$

Similarly we have:

$$\mathcal{M}_{\mathbb{P}^1}^d \cong \mathcal{M}_{\mathbb{P}^1}^d \quad \hat{\mathcal{M}}_{\mathbb{P}^1}^d = \hat{\mathcal{M}}_{\mathbb{P}^1}^d$$

$$\mu \searrow \quad \swarrow \hat{\mu}$$

Universal Higgs pair on $\mathcal{M}^d \times \mathbb{C}$

$$(\mathbb{P}E, \Phi) \longrightarrow \mathcal{M}_{\mathbb{P}^1}^d \times \mathbb{C}$$

$$\text{let } \Psi = \mathbb{P}E / \hat{\mathcal{M}}_{\mathbb{P}^1}^d \times \mathbb{C}$$

$\mathcal{B} =$ Gerbe of ~~$\mathbb{S}U_r$~~ liftings to SU_r bundles

$\Rightarrow \mathcal{B}$ is a \mathbb{Z}_r -gerbe.

\Rightarrow extend the structure to $U(1)$

Theorem 1. For any integers $d, e \in \mathbb{Z}$,

$$(\mathcal{M}_{\mathbb{P}^1}^d, \mathcal{B}^{\otimes e}) \xleftrightarrow{\text{SYZ}} (\hat{\mathcal{M}}_{\mathbb{P}^1}^e, \hat{\mathcal{B}}^{\otimes d})$$

$\hat{\mathcal{B}} : (\Gamma\text{-equivariant objects}) \Rightarrow (\text{objects on orbifold } \hat{\mu})$

Thm 2. $h_{st}^{p,q}(\mathcal{M}_{DR}^d, \mathcal{B}^e) = h_{st}^{p,q}(\hat{\mathcal{M}}_{DR}^e, \hat{\mathcal{B}}^{\otimes d})$

$\Gamma = 2$ and 3 (~~de~~) de cohomology to \mathcal{V} .

(Non-compact spaces — string-mixed Hodge $\#$'s.

→ dependence on the algebraic structure as opposed to
hds structure,

→ Cpt hyperkähler $\Rightarrow h^{p,q} = h^{q,p}$ (mirror to itself)
(orbifold) $\Rightarrow (h_{st})$