M. Thaddeus : Higgs bundles spaces as mirror partners 3/30/2002

w/ T. Hausel

$\mathcal{M}, \hat{\mathcal{M}}$

$h^{p,q}(\mathcal{M}) = h^{n-p,q}(\hat{\mathcal{M}})$

Defn. $\mathcal{M} \xleftarrow{SYZ} \hat{\mathcal{M}}$

if
$$\begin{array}{ccc} M & & \hat{M} \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\hat{\mu}} & \\ & N & \end{array}$$

$\dim_{\mathbb{R}} N = n$

s.t. for reg. value x of μ and $\hat{\mu}$,

$L_x = \mu^{-1}(x)$ and $\hat{L}_x = \hat{\mu}^{-1}(x)$

are special Lagrangian tori

$L_x \simeq (S^1)^n$

$\omega|_{L_x} = 0$

$\mathrm{Im}\, \Omega|_{L_x} = 0$

$L_x \cong \mathrm{Hom}(\pi_1(\hat{L}_x), U(1))$

$\hat{L}_x = \mathrm{Hom}(\pi_1(L_x), U(1))$

depend smoothly on x.

Lemma M-hyperkähler $L \subset M$ complex Lagrangian w.r.t J $\Rightarrow$ special Lagrangian w.r.t. J_2

Let C be a smooth curve genus $= g$

$\mathcal{M}_{DR} = \{\text{irreducible local systems } (SL_r)\}$

$\mathcal{M}_{Dol} = \{\text{stable } SL_r\text{-Higgs bundles}\}$

$= \{(E,\varphi) \mid \Lambda^r E \cong \mathcal{O},\ \mathrm{tr}\,\varphi = 0,\ \text{stable}\}$

$$\begin{tikzcd}
\mathcal{M}_{DR} \cong \mathcal{M}_{Dol} \arrow[dr, "\mu = \text{Hitchin}"] & & \mathcal{M}_{Dol}/\Gamma = \hat{\mathcal{M}}_{Dol} \cong \hat{\mathcal{M}}_{DR} \arrow[dl, "\hat{\mu}"] \\
& V &
\end{tikzcd}$$

$\mu^{-1}(x) = \mathrm{Prym} = \mathrm{Nm}^{-1}(0)$

norm map : $\mathrm{Nm} : \mathrm{Pic}^0(\tilde{C}) \to \mathrm{Pic}^0(C)$

$\hat{\mu}^{-1}(x) = ?$

$0 \to \mathrm{Prym} \to \mathrm{Pic}^0(\tilde{C}) \xrightarrow{\mathrm{Nm}} \mathrm{Pic}^0(C) \to 0$

$0 \to \mathrm{Pic}^0 C \xrightarrow{\pi^*} \mathrm{Pic}^0(\tilde{C}) \to \mathrm{Prym}/\Gamma \to 0$

$\Gamma = \mathrm{Prym} \cap \pi^* \mathrm{Pic}^0 C = \mathrm{Pic}^0 C[r] \Rightarrow \mathcal{M}_{Dol}/\Gamma$

$\hat{\mathcal{M}}_{DR} = \mathcal{M}_{DR}/\Gamma = \{\text{irred } PGL_r \text{ local systems}\}$

$\hat{\mathcal{M}}_{Dol} = \mathcal{M}_{Dol} / \Gamma = \{$ stable PGL_r Higgs bundles $\}$
(topological trivial)

$SL_r \longleftrightarrow PGL_r$ are dual (Langlands)

<u>Proposition</u> $\mathcal{M}_{DR} \overset{SYZ}{\longleftrightarrow} \hat{\mathcal{M}}_{DR}$

for any reductive group G and the dual group $\hat{G}$

(What is the fiber of μ : $(SL_r) \Rrightarrow$ Prym
$(G) \Rrightarrow Hom_W(\Lambda, Pic^\circ C)$
"cameral cover"
$(\check{G}) \Rrightarrow Hom_W(\hat{\Lambda}, Pic^\circ C)$!

Extend the def of mirror symmetry :
replace $\mathcal{M} \leftrightarrow (\mathcal{M}, B)$
$\hat{\mathcal{M}} \longleftrightarrow (\hat{\mathcal{M}}, \hat{B})$
where B & $\hat{B}$ are B-fields

Def of (Gerbe) : A - abelian grp

An (flat) <u>A - gerb B</u> on X :
is a <u>sheaf of categories</u> locally isomorphic to
the category of (flat) bundles w/ structure group A
(this "simplified" version is enough for our purposes)

E.g. Ψ = PGL - bundle on X

B = category of liftings of Ψ to SL_r bundle

Any two liftings differ by tensorization by a $\mathbb{Z}_r$ bundle

$\Rightarrow$ B is $\mathbb{Z}_r$ gerbe.

Reason:

$$0 \to \mathbb{Z}_r \to SL_r \to PGL_r \to 1$$

Thm (Giraud)

$$\left\{\begin{array}{l}\text{Isom classes of}\\ \text{flat } A \text{ gerbs}\\ \text{on } X\end{array}\right\} = H^2(X, A)$$

(B-fields) $\longleftrightarrow$ (flat Gerbes)

- An automorphism is an aut of sheaf of categories :
- Any aut. is a tensorization by an A-bundle
- 2 aut.'s are equivalent if the bundles are $\cong$
- Equiv classes of aut.'s $= H^1(X, A)$
- Trivialization is an iso to the trivial bundle
- Two triv's are equivalent if $f \circ g^{-1} \cong id$.

Define $\mathrm{Triv}(X, B) = \{$equiv cl. of triv's$\} =$
$=$ a torsor for $H^1(X, A)$
(over a point)

Def'n: $(\mathcal{M}, B) \xleftrightarrow{\text{SYZ}} (\hat{\mathcal{M}}, \hat{B})$

if
$$\begin{tikzcd} \mathcal{M} \arrow[dr] & & \hat{\mathcal{M}} \arrow[dl] \\ & N & \end{tikzcd}$$

$\mathcal{M}$ - Calabi - Yau
$B =$ flat $U(1)$ gerbe on $\mathcal{M}$

everything else is the same except:

- $L_x = \mathrm{Triv}(\hat{L}_x, \hat{B})$
- $\hat{L}_x = \mathrm{Triv}(L_x, B)$

(N. Hitchin: "Lectures on special Lagrangians in manifolds")

- depends smoothly on x

Define $\mathcal{M}^d_{Dol} = \{$ stable $(E, \varphi) \mid \Lambda^r E \cong \mathcal{O}(dc)$, $\operatorname{tr} \varphi = 0\}$
$c =$ base pt ~~of~~ in C

$\mathcal{M}^d_{DR} = \{$ irred. local systems on $C \setminus \{c\}$ $\cdots \}$

Γ acts (finite ab. grp.) $\hat{\mathcal{M}} = \mathcal{M}/\Gamma$ (DR, Dol)

Moduli of PGL_r Higgs $= \bigsqcup_{d=0}^{r} \hat{\mathcal{M}}^d_{Dol}$

Similarly we have:

$$\begin{array}{ccc} \mathcal{M}^d_{DR} \cong \mathcal{M}^d_{Dol} & & \hat{\mathcal{M}}^d_{Dol} = \hat{\mathcal{M}}^d_{DR} \\ & \searrow^{\mu} \quad \swarrow^{\hat{\mu}} & \\ & V & \end{array}$$

Universal Higgs pair on $\mathcal{M}^d \times C$

$$(\mathbb{P}E, \Phi) \longrightarrow \mathcal{M}^d_{Dol} \times C$$

Let $\Psi = \mathbb{P}E|_{\hat{\mathcal{M}}^d_{Dol} \times \{c\}}$

$B =$ Gerbe of ~~SL_r~~ liftings to SL_r bundles

$\Rightarrow B$ is a $\mathbb{Z}_r$-gerbe.

$\Rightarrow$ extend the structure to $U(1)$

Theorem 1. For any integers $d, e \in \mathbb{Z}$,

$$(\mathcal{M}^d_{DR}, B^{\otimes e}) \xleftrightarrow{SYZ} (\hat{\mathcal{M}}^e_{DR}, \hat{B}^{\otimes d})$$

$\hat{B}$: (Γ-equivariant objects) $\Rightarrow$ (objects on orbifold $\hat{\mathcal{M}}$)

<u>Thm 2.</u> $h^{p,q}_{st}(\mathcal{M}^d_{DR}, \mathcal{B}^{\otimes e}) = h^{p,q}_{st}(\hat{\mathcal{M}}^e_{DR}, \hat{B}^{\otimes d})$

$r = 2$ and 3 (d, e coprime to r.

(Non-compact spaces — Stringy ~~mixed~~ Hodge #'s.
$\Rightarrow$ dependence on the algebraic structure as opposed to holo structure,
$\rightarrow$ Cpct hyperkähler $\Rightarrow h^{p,q} = h^{n-p,q}$ (mirror to itself)
(orbifold) $\Rightarrow (h_{st})$